CMSC 22880 - Day 8

Looking Ahead
Today:
Static variables & methods in Java
Adding on to gates
Working up to:
Superdense codes
Quantum teleportation
Midterm: February 8th, in-person during class

The Same Entangle Circuit

The Quantum Register

- Quantum Registers hold collection of qubits used for computation, $q_0, q_1, \ldots, q_n$
  - One or more may be used in a quantum circuit
  - Qubits are initialized to the ground state $|0\rangle$
- Within circuit diagrams, each qubit in the register is represented with a wire
- Qubits in parallel are combined to form an overall state with the tensor product
  
  $q_0 \otimes q_1 \otimes \ldots \otimes q_n = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle = |00\ldots0\rangle$

Remember: when combining qubits:
Top to Bottom reads Left to Right

- How could we construct the the Opposite Entangle Circuit?
- What other types of entangled states can we create?
- What gates do we need if both qubits start in $|0\rangle$?
What is the equation to solve this?

a) \( M = (H \otimes \text{Identity}) \times \text{CNOT} \)

b) \( M = \text{CNOT} \times (H \otimes \text{Identity}) \)

c) \( M = (H \times \text{Identity}) \otimes \text{CNOT} \)

d) \( M = \text{CNOT} \otimes (H \times \text{Identity}) \)

Calculating Output with \(|00\rangle\):

\[ |\psi_{\text{out}}\rangle = M |\psi_{\text{in}}\rangle \]

\[ |\psi_{\text{out}}\rangle = \text{CNOT}(H \otimes \text{Identity}) |00\rangle \]

\[ |\psi_{\text{out}}\rangle = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \]

New Circuit: Add NOT gate

Additional NOT gate transforms bottom qubit to \(|1\rangle\) before entering entangling circuit.

What do you think will happen to the output qubit state?
New Circuit: NOT Bottom Qubit

\[
\left| \psi_{\text{out}} \right\rangle = M \left| \psi_{\text{in}} \right\rangle = CNOT(H \otimes \text{NOT}) \left| 00 \right\rangle
\]

This circuit created the opposite entangle state!

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1/\sqrt{2} & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1/\sqrt{2} & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1/\sqrt{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Adding NOT before H

\[
\left| \psi_{\text{out}} \right\rangle = M \left| \psi_{\text{in}} \right\rangle = CNOT(H \otimes \text{Identity})(\text{NOT} \otimes \text{Identity}) \left| 00 \right\rangle
\]

We created same entangle state... WITH PHASE!

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1/\sqrt{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1/\sqrt{2} & 1 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Entangled Two Qubit States

Different circuit variations produce same/opposite entangle with/without phase

- We just looked at a common example that uses H+CNOT as foundation....many other entangling circuits exist!
- Resulting states are Bell States

\[
\begin{align*}
\left| \Phi^+ \right\rangle &= \frac{1}{\sqrt{2}} (\left| 00 \right\rangle + \left| 11 \right\rangle) \\
\left| \Phi^- \right\rangle &= \frac{1}{\sqrt{2}} (\left| 00 \right\rangle - \left| 11 \right\rangle) \\
\left| \Psi^+ \right\rangle &= \frac{1}{\sqrt{2}} (\left| 01 \right\rangle + \left| 10 \right\rangle) \\
\left| \Psi^- \right\rangle &= \frac{1}{\sqrt{2}} (\left| 01 \right\rangle - \left| 10 \right\rangle)
\end{align*}
\]

New Circuit: Add NOT gate after Entangle

Additional NOT gate on top qubit after entangle.

What do you think will happen to the output qubit state?
NOT after Entangle

\[ |\psi_{\text{out}}\rangle = M |\psi_{\text{in}}\rangle \]
\[ |\psi_{\text{out}}\rangle = (\text{NOT} \otimes \text{Identity}) \text{CNOT} (H \otimes \text{Identity}) |00\rangle \]
\[ |\psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ |\psi_{\text{out}}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \]

This circuit also creates the opposite entangle state!

PRACTICE: What state results when |01\rangle is the input to this circuit?

A. \[ \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]
B. \[ \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \]
C. \[ \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \]
D. \[ \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \]

PRACTICE: Which matrix corresponds to circuit that produces two-qubit entangled states?

A. The left
B. The right
C. Both
D. Neither
PRACTICE: Which matrix corresponds to circuit that produces two-qubit entangled states?

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\quad \quad \quad \quad \quad
\frac{1}{\sqrt{2}}
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & -1 & 0 & 1 \\
-1 & 0 & 1 & 0
\end{bmatrix}
\]

A. The left  
B. The right  
C. Both  
D. Neither

Maximally-entangled, two qubit states are often referred to as:

a. Feynman States  
b. CNOT States  
c. Bell States  
d. Q - States

(True / False) The only way to entangle two qubits is with a CNOT and a H gate.

Determine the quantum state that results from the following circuit:

\[
|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[
|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}
\begin{bmatrix}
1 \\
0 \\
0 \\
-1
\end{bmatrix}
\]

\[
|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}
\begin{bmatrix}
0 \\
1 \\
1 \\
0
\end{bmatrix}
\]

\[
|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}
\begin{bmatrix}
0 \\
1 \\
-1 \\
0
\end{bmatrix}
\]

(True / False) The output qubits of this quantum circuit are entangled.
Inverting CNOT and CZ

Quantum Circuit Structure

In a circuit, order and orientation of gates matters!

- What happens to the matrix when you flip the orientation of CNOT?

Review: The CNOT Gate

CNOT acts on two qubits: a control and a target. The input state is expressed as:

\[
|\psi_{\text{in}}\rangle \otimes |\phi_{\text{in}}\rangle = |\psi_{\text{out}}\rangle \otimes |\phi_{\text{out}}\rangle
\]

CNOT Symbol

CNOT Matrix

CNOS State Transformation

State Evolution:

\[
|0\rangle \otimes |0\rangle = |00\rangle \Rightarrow |0\rangle \otimes |0\rangle = |00\rangle
\]

\[
|0\rangle \otimes |1\rangle = |01\rangle \Rightarrow |0\rangle \otimes |1\rangle = |01\rangle
\]

\[
|1\rangle \otimes |0\rangle = |10\rangle \Rightarrow |1\rangle \otimes |0\rangle = |10\rangle
\]

\[
|1\rangle \otimes |1\rangle = |11\rangle \Rightarrow |1\rangle \otimes |0\rangle = |10\rangle
\]
Invert the CNOT Gate

Quantum operations may be inverted in a circuit. What happens to their matrix?

- Let’s analyze input and output relations for the inverted CNOT!

CZ State Transformation

CZ acts on two qubits, a control and a target, selectively adding relative phase

Invert the CNOT Gate

Invert the CNOT

What happens when CZ is inverted?

Inverted CZ

Inverted CZ has the same matrix!

CZ State Evolution:

|control⟩ ⊗ |target⟩ ↦ |target⟩ ⊗ |control⟩

CZ Matrix

Inverted CZ Matrix

|target⟩ ⊗ |control⟩ ↦ |target⟩ ⊗ |control⟩

CZ Symbol

CZ Matrix
Summary

- Just like order matters for combining gates and qubits, orientation is important for multi-qubit operations
  - CNOT gate
- The CZ gate is special because inverting control and target results in the same qubit transformation and matrix

Controlled Gates and the CCNOT/Toffoli

Controlled Gates

Think about how the NOT gate becomes the CNOT gate

Examine the CNOT matrix. It contains:
1) An Identity Gate
2) An NOT Gate

Controlled Gates

- Creating a controlled gate from an arbitrary quantum gate U is possible.
- Add a second control qubit, and only apply gate if control has a probability amplitude for |1⟩ (remember: control can be in superposition!):
Example: Controlled Hadamard

\[ |10⟩ \]

\[ \frac{1}{\sqrt{2}} (|0⟩ + |1⟩) \]

\[ \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \sqrt{2} & -\sqrt{2} \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
1 & 0 & 0 & 0
\end{bmatrix} \]

\[ |1⟩ \]

\[ \frac{1}{\sqrt{2}} [1 \quad 1 \quad -1] \]

State Calculation:

\[ (C-H)|10⟩ = \frac{1}{\sqrt{2}} (|0⟩ + |1⟩) \]

Additional Controls on CNOT

- The Controlled-CNOT, CCNOT, is also called the Toffoli gate.
- The Toffoli acts on three qubits: two controls and one target.
  - NOT operation on target if and only if both controls have a probability amplitude for |1⟩.

\[ \text{Two-qubit Identity Matrix, } \left[ \begin{array}{cc}
0 & 0 \\
0 & 1
\end{array} \right] \]

\[ \text{CNOT Matrix, } \left[ \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array} \right] \]

PRACTICE: What is the output corresponding to the input \( |100⟩ \) for the pictured quantum circuit?

a. \( |101⟩ \)  
b. \( |100⟩ \)  
c. \( |110⟩ \)  
d. \( |111⟩ \)
PRACTICE: What is the output corresponding to the input $|100\rangle$ for the pictured quantum circuit?

- a. $|101\rangle$
- b. $|100\rangle$
- c. $|110\rangle$
- d. $|111\rangle$

1) Consider the quantum circuit pictured here. If the input is $|110\rangle$, what is the output?

- a. $|001\rangle$
- b. $|000\rangle$
- c. $|100\rangle$
- d. $|111\rangle$

2) Consider the quantum circuit pictured here. If the input is $|110\rangle$, what is the output?

- a. $|100\rangle$
- b. $|101\rangle$
- c. $|110\rangle$
- d. $|111\rangle$

3) Which matrix represents a controlled-CZ operation?

- a. \[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
- b. \[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
- c. \[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
- d. \[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]