# Graph Theory

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Spring Quarter, 2014

**Course Homepage:**

http://www.classes.cs.uchicago.edu/archive/2014/spring/27500-1/GTspring14.html

**You are encouraged to work together on solving homework problems, but please put the names of your collaborators clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class.**

## Homework 1, due April 10

1. **(BM 1.2.17 p.19, Edge-Transitive Graph)** A simple graph is *edge-transitive* if, for any two edges $uv$ and $xy$, there is an automorphism $\alpha$ such that $\alpha(u)\alpha(v) = xy$.

   a) Show that *Petersen graph* (see Figure 1) is edge-transitive.

   b) Vertices $u$ and $v$ in a graph are called *similar* if there is an automorphism $\alpha$ such that $\alpha(u) = v$. Graphs in which all vertices are similar are called *vertex-transitive*. Find a graph which is vertex-transitive but not edge-transitive.

   c) Show that any graph without isolated vertices (i.e. vertices of degree 0) which is edge-transitive but not vertex-transitive is bipartite.

2. **(BM 1.2.19 p.20, Generalized Petersen Graph)** Let $k$ and $n$ be positive integers, with $n > 2k$. The generalized Petersen graph $P_{k,n}$ is the simple graph with vertices $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$, and edges $x_i x_{i+1}, y_i y_{i+k}, x_i y_i$, $1 \leq i \leq n$, indices being taken modulo $n$. (Note that $P_{2,5}$ is the Petersen graph.)

   a) Draw the graphs $P_{2,7}$ and $P_{3,8}$.

   b) Which of these two graphs are vertex-transitive, and which are edge-transitive?

3. **(BM 1.2.20 p.20)** Recall that a (right) *eigenvector* of a square matrix $\mathbf{B}$ is a non-zero vector $v$ such that $\mathbf{B}v = \lambda v$ for some real or complex number $\lambda$. This number $\lambda$ is called the *eigenvalue* of $\mathbf{B}$ corresponding to eigenvector $v$. The *adjacency matrix* of a graph $G$ with $n$ vertices is the $n \times n$ matrix $\mathbf{A} := (a_{uv})$, where $a_{uv}$ is the number of edges joining vertices $u$ and $v$, each loop counting as two edges. Show that if $G$ is simple and the eigenvalues of $\mathbf{A}$ are distinct, then every automorphism of $G$ is of order one or two.

4. **(BM 1.3.9 p.25)** The *incidence matrix* of a graph $G$ with $n$ vertices and $m$ edges is the $n \times m$ matrix $\mathbf{M} := (m_{ve})$, where $m_{ve}$ is the number of times (0, 1, or 2) that vertex $v$ and edge $e$ are incident. The *line graph* of an undirected graph $G$ is another graph $L(G)$ that has as vertices the edges of $G$, two edges being adjacent if they have an end in common. An *eigenvalue* of a graph is an eigenvalue of its adjacency matrix. Let $G$ be a simple graph with incidence matrix $\mathbf{M}$.

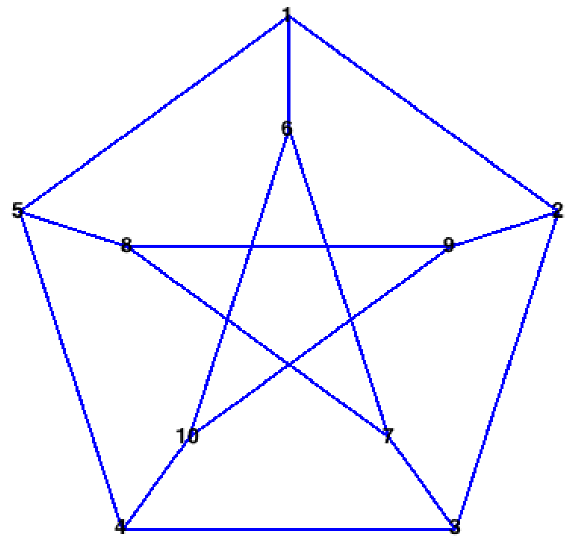


Figure 1: Petersen graph.

a) Show that the adjacency matrix of its line graph $L(G)$ is $\mathbf{M^tM} - 2\mathbf{I}$, where $\mathbf{I}$ is the $m \times m$ identity matrix.

b) Recall that a symmetric matrix $B \in S\mathbb{R}^{n \times n}$ is called *positive-semidefinite* if $x^T Bx \geq 0$ for all $x \in \mathbb{R}^n$. Using the fact that $\mathbf{M^tM}$ is positive-semidefinite, deduce that:

**i)** each eigenvalue of $L(G)$ is at least $-2$,

**ii)** if the rank of $\mathbf{M}$ is less than $m$, then $-2$ is an eigenvalue of $L(G)$.

**5. (BM 1.3.12 p.26, Sperner's Lemma)** Let $T$ be a triangle in the plane. A subdivision of $T$ into triangles is *simplicial* if any two of the triangles which intersect have either a vertex or an edge in common. Consider an arbitrary simplicial subdivision of $T$ into triangles. Assign the colours red, blue, and green to the vertices of these triangles in such a way that each colour is missing from one side of $T$ but appears on the other two sides. (Thus, in particular, the vertices of T are assigned the colours red, blue, and green in some order.)

a) Show that the number of triangles in the subdivision whose vertices receive all three colours is odd.

b) Deduce that there is always at least one such triangle.

**6. (BM 1.3.15 p.27, The de Bruijn-Erdős Theorem)**

**a)** Let $G[X, Y]$ be a bipartite graph, each vertex of which is joined to at least one, but not all, vertices in the other part. Suppose that $d(x) \geq d(y)$ for all $xy \notin E$. Show that $|Y| \geq |X|$, with equality if and only if $d(x) = d(y)$ for all $xy \notin E$ with $x \in X$ and $y \in Y$.

**b)** Recall that a *finite projective plane* is a geometric configuration $(P, \mathcal{L})$ in which:

**i)** any two points lie on exactly one line,

**ii)** any two lines meet in exactly one point,

**iii)** there are four points no three of which lie on a line.

(Condition (iii) serves only to exclude two trivial configurations – the *pencil*, in which all points are collinear, and the *near-pencil*, in which all but one of the points are collinear.) Deduce the following theorem.

Let $(P, \mathcal{L})$ be a geometric configuration in which any two points lie on exactly one line and not all points lie on a single line. Then $|\mathcal{L}| \geq |P|$. Furthermore, if $|\mathcal{L}| = |P|$, then $(P, \mathcal{L})$ is either a finite projective plane or a near-pencil.