CS281 Spring 2010: Homework 6

Due Wednesday, May 26th – SOLUTION

1. **(10 points)**
   The problem $4TA-SAT$ is defined as follows: Given a boolean expression E, does E have at least four satisfying truth assignments? Show that $4TA-SAT$ is NP-complete.

   Sol'n. Clearly $4TA-SAT$ is in $\mathcal{N}P$ as a NTM can guess 4 assignments and verify them in polynomial time. It remains to show that $4TA-SAT$ is NP-hard. To do so we may reduce SAT, a known NP-complete problem to $4TA-SAT$. Let F be an instance of SAT, namely a Boolean formula. Then let z_1, z_2 be two variables which do not appear in F. Then the Boolean formula $E = F \wedge (z_1 \vee \bar{z}_1 \vee z_2 \vee \bar{z}_2)$ has a truth assignment iff F has a truth assignment (since the last clause is always true), and moreover E has a truth assignment iff it has at least four, since we may assign the variables z_1, z_2 arbitrarily. So $F \mapsto E$ defines a poly-time reduction of $4TA-SAT$ to SAT.

2. **(10 Points)**
   Give a polynomial time algorithm to solve the problem 2-SAT, i.e., satisfiability for CNF boolean expressions with only two literals per clause.
   Hint: If one of the two literals in a clause is false, the other is forced to be true. Start with an assumption about the truth of one variable and then chase down all the consequences for all the other variables.

   Sol'n. The easiest way to do this is to reduce to the problem of (n-fold) reachability in a digraph. This approach is described on pp.7-17 of `www.cs.tau.ac.il/ safra/Complexity/2SAT.ppt`. Otherwise you must carefully present an algorithm (that tries assignments and tracks their consequences in an organized, efficient manner) and give its analysis in full detail to prove it has polynomial run-time.

3. **(10 Points)**
   Consider a family of 3-CNF expressions. The expression E_n has n variables, $x_1, x_2 \ldots x_n$. For each set of three distinct integers between 1 and n, say i, j and k, E_n has clauses $(x_i + x_j + x_k)$ and $(\bar{x}_i + \bar{x}_j + \bar{x}_k)$. Is E_n satisfiable for $n = 4$?

   Sol'n. For $n = 4$ there are only four variables. Set two to true and two to false. Then each clause must contain at least one true variable and one false, therefore regardless of whether the clause's literals are variables or variable negations there will be a true literal in each clause, hence each clause is true and so their conjunction is as well.