Solutions of Midsem (Graph Theory)

1. (5) State the meaning of blocks and block tree, and the properties of block decompositions of a graph.

   Block: Maximal nonseparable subgraph.
   Block tree: The tree defined by the blocks and the separating vertices with edges between a block B and a separating vertex v iff $v \in B$.
   Properties: Two blocks intersect at most one vertex, which is a separating vertex. Every cycle is included totally within a block of the decomposition.

2. (5) State the meaning of ear decomposition for undirected graphs. Which undirected graphs have ear decomposition?

   Ear decomposition: An ear decomposition of a graph G is a sequence of graphs $G_1, G_2, \ldots, G_k$ such that G_1 is a cycle $G_n = G$ and for each $i = 1, 2, \ldots, k-1$, G_{i+1} can be obtained from G_i by adding an ear. The only undirected graphs to have ear decomposition are the nonseparable graphs.

3. (10) Let G be a connected graph and let S be a set of edges of G. Show that the following statements are equivalent.

   (a) S is a spanning tree of G.

   (b) S contains no cycle of G and is maximal with respect to this property.

   (c) S meets every bond of G and is minimal with respect to this property.

   (a) ⇒ (b): Since S is a tree it does not contain any cycle of G. Furthermore, adding any edge into S creates a cycle, namely a fundamental cycle. Thus, S is a maximal subgraph that contains no cycle.

   (b) ⇒ (a): If $|S| \geq n$ then S contains a cycle. Assume that S is not connected; there exists an edge $e \in E(G)$ that connects two connected components X and Y of S. Let $S' = S \cup \{e\}$. S' does not contain a cycle since S does not contain a cycle and e is an edge cut of S'; i.e. there cannot be a cycle in S' containing e. By maximality of S we have a contradiction; S must be connected. Furthermore, if S is not spanning, there exists a vertex v not covered by S and adding any of the edges incident to v into S would not create cycles. Again, by maximality, S must be spanning. We already know that S is connected and of size at most $n-1$, this implies that it is a spanning tree of G.

   (a) ⇒ (c): Since S is connected, it meets every bond of G. Furthermore, if we remove an edge from S, we disconnect it into two connected components X and Y. Since the new subgraph does not meet the edge cut $\partial(X)$, we conclude that S is minimal.

   (c) ⇒ (a): If S meets every bond of G, S must be connected and spanning. Assume that S is not connected, then the vertices of G can be partitioned into two sets X and Y such that S does not include an edge of the edge cut $\partial(X)$; contradiction. To show that S is spanning, again assume otherwise and let X be the set of vertices that S covers. Again, $S \cap \partial(X) = \emptyset$. Therefore, S is spanning and connected, furthermore, any spanning tree T of S is a spanning tree of G and T meets every bond of G as proven above. By minimality, S must be a spanning tree of G.

4. (10) Let $G = (V, E)$ be a graph and let $\cong$ denote the binary relation on E defined by: $e \cong f$ iff either $e = f$ or there is a bond of G containing both e and f. Show that:

   (a) The relation $\cong$ is an equivalence relation on E.

   (b) The subgraphs of G induced by the equivalence classes under this relation are its nontrivial blocks.

   (a) Clearly, the relation $\cong$ is symmetric. Now, let $e \cong f$ and $f \cong g$. Then, by definition, there is a bond B containing e and f and a bond C containing f and g. If $g \in B$ or $e \in C$ then $e \cong g$. Otherwise, consider the symmetric difference $D = B \triangle C$. We know that D is an edge cut and it includes both e and g; therefore, D is a disjoint union of bonds. Furthermore, it can be shown that the bond in this disjoint union containing e also contains g, hence $e \cong g$. This proves that $\cong$ is an equivalence relation.

   (b) Pick a vertex v of a block F of G and consider the set $B_v(F) = \partial(v) \cap F$. Removing $B_v(F)$ from the graph disconnects v from the block F and $B_v(F)$ is minimal with respect to this property. Therefore $B_v(F)$ is an edge cut and by minimality, $B_v(F)$ is in fact a bond. For $v, u \in F$, the bonds $B_v(F)$ and $B_u(F)$ intersect iff there exists en edge between v and u. Any of these bonds is included in an equivalence class and they are in the same class iff two bonds intersect. Since the intersection of the bonds is equivalent to the adjacency of the vertices of F the bonds defined by the vertices of F cover the whole block F. What remains to prove is that any two edges e, f that are in different blocks cannot be equivalent, i.e. there is no bond including both e and f. Assume towards a contradiction that $e \in F_1$ and $f \in F_2$ are both included in a bond B. Since $B \cap F_1$ is a proper subset of B, it cannot be an edge cut. Therefore, there exists a spanning tree for the block F_1 that does not include any edges from B. Similarly, there exists a spanning tree for all the other blocks free of the edges of B. Then the union of these trees is a tree spanning the whole graph G which implies that G B is connected; contradiction.