1a \[ M_0 = (q.g\left(f^2\right))((2y, z, 2x, (a + u + x)) x (2x, f, y)) \]
\[ M_1 = 2y. \bar{x}(2z, 2x, x(2z, z, x)) \]

1b \( q / F^3 M_0 = \)
\[ (q^b, q^b(q^2))((2y, z, 2x, (a + u + x)) x (2x, f, y)) \]
\( \alpha \)-conversion before subst.

\( F / g^3 M_0 = M_0 \) - no change, \( g \notin FV(M_0) \)
\( FV(M_0) = \{ f, x, y \} \)

2. \( \forall \Gamma, \forall e, \Gamma \vdash e \text{ ok} \iff \text{FV}(e) \subseteq \Gamma \)

(\( \Rightarrow \)) \( \forall \Gamma, \forall e, \Gamma \vdash e \text{ ok} \Rightarrow \text{FV}(e) \subseteq \Gamma \).

Assume, for some \( \Gamma, e \), that \( \Gamma \vdash e \text{ ok} \). We prove \( \text{FV}(e) \subseteq \Gamma \) by ind. on derivation of \( \Gamma \vdash e \text{ ok} \).

Base Case 1: \( \Gamma \vdash e \text{ ok} \) by rule S1 (the Var rule).
Then \( e = x \) for some variable \( x \), and \( x \in \Gamma \).
Then \( \text{FV}(e) = \{ x \} \) (defn of FV), and \( \{ x \} \subseteq \Gamma \).

Base Case 2: \( \Gamma \vdash e \text{ ok} \) by rule S2 (the Num rule).
Then \( e = n \) for some \( n \in \mathbb{N} \). Thus \( \text{FV}(e) = \text{FV}(n) = \emptyset \subseteq \Gamma \).

Ind Case 1: \( \Gamma \vdash e \text{ ok} \) by rule S3 (Plus rule).
Then \( e = + (e_1, e_2) \). By inversion of the rule, we have (1) \( \Gamma \vdash e_1 \text{ ok} \) and (2) \( \Gamma \vdash e_2 \text{ ok} \).
The Ind. Hypothesis is:

(IH1) $\Gamma \vdash e_1 \text{ ok} \implies \text{FV}(e_1) \subseteq \Gamma$  \hspace{1cm} (same $\Gamma$)

& (IH2) $\Gamma \vdash e_2 \text{ ok} \implies \text{FV}(e_2) \subseteq \Gamma$

By (1) and (IH1), we have (3) $\text{FV}(e_1) \subseteq \Gamma$ and

(4) $\text{FV}(e_2) \subseteq \Gamma$. Thus

$$\text{FV}(e) = \text{FV}(e_1) \cup \text{FV}(e_2) \subseteq \Gamma \quad \square$$

The case for the times rule is similar. (Ind. Case 2)

Ind Case 3: $\Gamma \vdash e \text{ ok}$ by rule $\text{S5}$ (Let rule).

Then $e = \text{let}(e_1, x, e_2)$. By inversion of the Let rule we have

(1) $\Gamma \vdash e_1 \text{ ok}$

(2) $\Gamma \cup \{x\} \vdash e_2 \text{ ok}$

The ind. hyp. is

(IH1) $\Gamma \vdash e_1 \text{ ok} \implies \text{FV}(e_1) \subseteq \Gamma$

& (IH2) $\Gamma \cup \{x\} \vdash e_2 \text{ ok} \implies \text{FV}(e_2) \subseteq \Gamma \cup \{x\}$

Then by (1) and (IH1), and (2) and (IH2) resp.

(3) $\text{FV}(e_1) \subseteq \Gamma$

(4) $\text{FV}(e_2) \subseteq \Gamma \cup \{x\}$ \hspace{1cm} ($\implies \text{FV}(e_2) \setminus \{x\} \subseteq \Gamma$)

which implies

(5) $\text{FV}(e) = \text{FV}(\text{let}(e_1, x, e_2))$

$$= \text{FV}(e_1) \cup (\text{FV}(e_2) \setminus \{x\})$$

$$\subseteq \Gamma$$ \hspace{1cm} \blacksquare$$
2. \(\forall e.\forall \Gamma . \text{FV}(e) \subseteq \Gamma \Rightarrow \Gamma \vdash e \text{ ok}\)

Assume \(\text{FV}(e) \subseteq \Gamma\). Prove \(\Gamma \vdash e \text{ ok}\) by ind. on structure of \(e\).

Base Case 1: \(e = x\), a variable.

Then \(\text{FV}(e) = \{x\}\), so \(\{x\} \subseteq \Gamma\) means \(x \in \Gamma\). Thus by rule S1, \(\Gamma \vdash e \text{ ok}\).

Base Case 2: \(e = n\), a number. Then \(\Gamma \vdash e \text{ ok}\) by rule Num.

Ind. Case 1: \(e = \text{plus}(e_1, e_2)\). Then \(\text{FV}(e) = \text{FV}(e_1) \cup \text{FV}(e_2)\).

So \(\text{FV}(e) \subseteq \Gamma \Rightarrow\):

1. \(\text{FV}(e_1) \subseteq \Gamma\)
2. \(\text{FV}(e_2) \subseteq \Gamma\)

The Ind. Hyp. is:

(IH1) \(\forall \Gamma'. \text{FV}(e_1) \subseteq \Gamma' \Rightarrow \Gamma' \vdash e_1 \text{ ok}\)

& (IH2) \(\forall \Gamma'. \text{FV}(e_2) \subseteq \Gamma' \Rightarrow \Gamma' \vdash e_2 \text{ ok}\)

Then

3. \(\Gamma \vdash e_1 \text{ ok}\) by (1) and (IH1)
4. \(\Gamma \vdash e_2 \text{ ok}\) by (2) and (IH2)

Then by (3) and (4) and rule Plus (S3), we have \(\Gamma \vdash e \text{ ok}\).

The Times case is similar. (Ind Case 2)

Ind Case 3: \(e = \text{let}(e_1, x . e_2)\).

Then \(\text{FV}(e) = \text{FV}(e_1) \cup (\text{FV}(e_2) \setminus \{x\})\), \(\text{FV}(e) \subseteq \Gamma\)

implies

1. \(\text{FV}(e_1) \subseteq \Gamma\)
2. \(\text{FV}(e_2) \setminus \{x\} \subseteq \Gamma\) \(\Rightarrow \text{FV}(e_2) \subseteq \Gamma \cup \{x\}\)
The Ind Hyp is

(IH1) \( \forall \Gamma'. \ FV(e_1) \in \Gamma' \Rightarrow \Gamma' \vdash e_1 \) ok

& (IH2) \( \forall \Gamma'. \ FV(e_2) \in \Gamma' \Rightarrow \Gamma' \vdash e_2 \) ok

Then

(3) \( \Gamma \vdash e_1 \) ok by (1) and (IH1)

(4) \( \Gamma \upharpoonright \{x\} \vdash e_2 \) ok by (2) and (IH2)

Then (3) and (4) and the Let rule give \( \Gamma \vdash e \) ok. \( \square \)
3. \( e \rightarrow e_1 \text{ and } e \rightarrow e_2 \Rightarrow e_1 = e_2 \).

Proof by induction on the derivation of \( e \rightarrow e_1 \) (i.e. by rule induction on the rules for \( \rightarrow \)).

**Base Case 1.** \( e \rightarrow e_1 \) by \( E_{10} \), the plus instruction.

Then \( e = \text{plus}(m,n) \) and \( e_1 = p \) where \( p = m+n \).

Then the only rule by which \( e \rightarrow e_2 \) is \( E_{10} \), so \( e_2 \) must be \( p \) as well. Hence \( e_1 = p = e_2 \). \( \square \)

Similarly for \( E_{11} \).

**Base Case 2.** \( e \rightarrow e_1 \) by \( E_{12} \), the let instruction.

Then \( e = \text{let}(n,x.e') \) and \( e_1 = \text{let}(n/x^{e'}) \).

The only rule that matches \( e \) is \( E_{12} \), so \( e \rightarrow e_2 \) by the same rule, so \( e_2 = \text{let}(n/x^{e'}) = e_1 \). \( \square \)

**Ind. Case 1.** \( e \rightarrow e_1 \) by \( E_{14} \), the left plus search rule.

Then \( e = \text{plus}(e_3, e_4) \) and \( e_1 = \text{plus}(e_3', e_4') \) where \( e_3 \rightarrow e_3' \). The induction hypothesis is:

\[ \text{(IH)} \forall f_1, f_2. \ e_3 \rightarrow f_1 \text{ and } e_3 \rightarrow f_2 \Rightarrow f_1 = f_2 \]

The only rule by which \( e \rightarrow e_2 \) can be deduced is \( E_{14} \), so \( \exists e_5. \ e_2 = \text{plus}(e_5, e_4) \), and \( e_3 \rightarrow e_5 \).

But by the (IH), taking \( f_1 = e_3' \) and \( f_2 = e_5 \), we have \( e_3' = e_5 \). But then \( e_2 = \text{plus}(e_5, e_4) = \text{plus}(e_3', e_4) = e_1 \). \( \square \)

All the other search rule cases follow the same pattern, with minor, and obvious changes.