CMSC-37110 Discrete Mathematics
SECOND MIDTERM EXAM November 14, 2006

This exam contributes 16% to your course grade.

Do **not** *use book or notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (5 points) We wish to distribute k identical coins between n children so that each of them receives at least 2 coins. Count the possible outcomes. Your answer should be a binomial coefficient. Prove.

2. (3+8 points) We roll n dice and obtain a string of length n over the alphabet $[6]$. Let X be the number of consecutive occurrences of 5 and 6 (in this order). (E. g., if the string is 53656361565 then $X = 2$.) Calculate (a) $E(X)$ (b) $\mathrm{Var}(X)$. Make your expressions simple. **Define your variables.**

3. (3+3+6 points)

   (a) State Fermat's little Theorem as stated in class. (Do not confuse it with the more general Euler-Fermat Theorem.) Your statement should be a correctly quantified formula involving no English words other than logical connectives (IF, AND, etc.) and the word PRIME.

   (b) Consider the following statement:
   $(\forall p)(\forall z)(p \text{ PRIME} \Rightarrow z^p \equiv z \pmod p)$.
   Prove that this statement is equivalent to Fermat's little Theorem.

   (c) Prove the statement under (b) by induction on z.

4. (4 points) Count the Hamilton cycles in the complete bipartite graph $K_{r,r}$.

5. (8+3 points) (a) Give a closed form expression for the number A_n of those n-digit integers which have only odd digits $(1, 3, 5, 7, 9)$ and all the five odd digits actually occur. (b) Asymptotically evaluate A_n, i. e., give a very simple expression which is asymptotically equal to A_n. Prove.

6. (7 points) According to a homework problem, every planar graph has a vertex of degree ≤ 5. Use this fact to prove that every planar graph is 6-colorable.

7. (6 points) Let p be a prime. Suppose p^k divides $n!$ (n-factorial). Prove: $k < \frac{n}{p-1}$.

8. (6+2B points) Prove that the 5×5 toroidal grid ("chessboard") is not planar. (2 bonus points for an elegant solution.)

9. (5B points: bonus problem) Prove that the knight's graph and the rook's graph on the 5×5 toroidal chessboard are isomorphic.

10. (8B points: bonus problem) Prove: for almost all graphs G, $\chi(G) > \omega(G)^{100}$. ($\chi(G)$ denotes the chromatic number and $\omega(G)$ is the clique number, i. e., the size of the largest clique (complete subgraph) in G.)