CMSC-37110 Discrete Mathematics
FIRST MIDTERM EXAM October 12, 2006

This exam contributes 16% to your course grade.

Do **not** *use book or notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as g.c.d. or binomial coefficients. **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (6 points) Prove: $(\forall x)(\exists y)(y > x \text{ and } x^{y-1} \equiv 1 \pmod{y})$. ($x$ and y are integers.)

2. (4 points) Let $a_1, a_2, \dots$ be a sequence of real numbers. Give an exact definition of the relation $\lim_{n\to\infty} a_n = 1$. Your definition should be a properly quantified formula and must not contain any English words except logical connectives such as "AND," "IF," "THEN," etc.

3. (2 points) Solve the equation $\binom{x}{2} = 5$. (x is a real number.)

4. (4 points) Let A be an n-set (set of n elements). What is the number of relations on A? (Recall that a relation on A is a subset of $A \times A$.)

5. (3 points) Count the 4-digit integers with increasing digits. (E.g. 2357 counts, but 2337 or 3454 does not count.) Evaluate your answer as an explicit integer. Do not use calculator; show every step of your calculation.

6. (5 points) Let p be an odd prime number. Prove: 4 is not a primitive root mod p. (Recall that the integer g is a primitive root mod p if $\gcd(g, p) = 1$ and $j = p - 1$ is the smallest positive number such that $g^j \equiv 1 \pmod{p}$.)

7. (5 points) Decide whether or not the following system of congruences is solvable. Prove your answer. If solvable, you don't need to solve the system, just prove that it is solvable.

   $x \equiv 7 \pmod{18}$

   $x \equiv 5 \pmod{12}$

   $x \equiv 1 \pmod{5}$

8. (2+3+B4 points) Evaluate each of the following sums as a closed-form expression (no summation symbols or dot-dot-dots). (a) $\sum_{k=0}^{n} 2^{-k}$.

   (b) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^{-k}$.

   (c) (BONUS PROBLEM) Find the largest term in sum (b).

9. (B5 points) (BONUS PROBLEM) Find a closed form expression for the sum $\sum_{k=0}^{\infty} \binom{n}{4k}$.

10. (B6 points) (BONUS PROBLEM) Prove: if a and b are relatively prime then $a+b$ divides $\binom{a+b}{a}$.