CMSC-27100 Discrete Mathematics
FINAL EXAM December 8, 2004

This exam contributes 30% to your course grade.

Do **not** *use book or notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as g.c.d. **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.**

1. (6+6 points) Consider the following two congruences.

   $(*)$ $21x \equiv 35 \pmod{60}$

   $(**)$ $12x \equiv 20 \pmod{240}$

   Decide whether or not each of the following holds:

   (a) $(\forall x)((*) \Rightarrow (**))$;

   (b) $(\forall x)((**) \Rightarrow (*))$.

   Prove your answers.

2. (10 points) Let t be a positive integer; assume $t \equiv 2 \pmod 7$. Determine the multiplicative inverse of 7 mod t. Your answer should be a simple formula denoting a number between 1 and t. Show how you obtained the formula. Verify that your answer is correct.

3. (8 points) Let p be a prime number and b an arbitrary integer. Prove: if $b^{p+1} \equiv 1 \pmod p$ then $b \equiv \pm 1 \pmod p$.

4. (2+2+4 points)

   (a) Define the big-Oh relation: for two sequences $\{a_n\}$ and $\{b_n\}$, we say that $a_n = O(b_n)$ if and only if ... – finish the sentence. Give a properly quantified formula, no English words.

   (b) Define the big-Omega relation: for two sequences $\{a_n\}$ and $\{b_n\}$, we say that $a_n = \Omega(b_n)$ if and only if ... – finish the sentence. Give a properly quantified formula, no English words.

   (c) Give an example of two sequences, $\{a_n\}$ and $\{b_n\}$, such that both sequences go to infinity, but $a_n \neq O(b_n)$ and $a_n \neq \Omega(b_n)$. Just state the sequences; you do not need to prove the correctness of your example (as long as it is correct).

5. (3+3+3+3+3 points)

   (a) Define the relation $a_n \gtrsim b_n$.

   (b) Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers. Assume $a_n \gtrsim b_n$. Does it follow that

i. $a_n = O(b_n)$

ii. $(\exists n_0)(\forall n \geq n_0)(a_n \geq b_n)$

iii. $(\exists n_0)(\forall n \geq n_0)(a_n \geq 0.9b_n)$

Prove your answers. The proof of a "no" answer is a counterexample.

6. (10 points) Consider a graph G with n vertices. Prove: if both G and its complement are planar then $n \leq 10$. You may use a result proved in class; clearly state the result.

7. (5 points) What is the probability that a hand of 10 cards includes at least one card of each of the four suits? Give a formula; do not evaluate. (There are 13 cards of each suit. The 10 cards are selected at random.) Name the method used.

8. (1+8+6+8+6 points) Recall that a tournament is an oriented complete graph. To obtain a *random tournament* on n vertices, we take K_n and flip a coin for each pair of vertices to decide which way to orient the edge between them. Let $T = (V, E)$ be a random tournament on n vertices.

   (a) State the size of the sample space for this experiment.

   (b) Let X denote the number of directed cycles of length 3 in T. Calculate $E(X)$. Your answer should be a simple formula. Prove your answer. Give a careful and clear definition of the random variables used. This definition accounts for half the credit.

   (c) Let A and B be two distinct (but not necessarily disjoint!) subsets of V; let $|A| = |B| = 3$. Let $C(A)$ denote the event that the subdigraph of T induced on A is a directed cycle (of length 3); the event $C(B)$ is defined analogously. Prove that $C(A)$ and $C(B)$ are independent.

   (d) Determine the standard deviation of X. Prove your answer.

   (e) Prove the Weak Law of Large Numbers for X. In other words, for $\epsilon > 0$, let $p_n(\epsilon)$ denote the probability that $|X - E(X)| \geq \epsilon E(X)$. Prove: $(\forall \epsilon > 0)(\lim_{n\to\infty} p_n(\epsilon) = 0)$.

9. (8 points) Prove the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

10. (6+4+4 points) Recall that the generating function of a sequence $\{a_n\}$ is the power series $\sum_{n=0}^{\infty} a_n x^n$. Give a closed-form expression for the generating function of each of the following sequences:

(a) $a_n = n^2$;

(b) $b_n = b_{n-1} + 6b_{n-2}$ $(n \geq 2)$ and $b_0 = 3$, $b_1 = 1$;

(c) $c_n = 1$ if $n \equiv 4 \pmod 7$ and $c_n = 0$ otherwise $(n = 0, 1, \dots)$.

11. (3+10 points)

 (a) Define what is a stationary state of a finite Markov chain.

 (b) Find a finite Markov chain which has a unique stationary state but this state is not everywhere positive (not every vertex has a positive stationary probability). Describe your Markov chain both in diagram and by the transition matrix. Make your Markov chain as small as possible. Prove your answer (in particular, prove the uniqueness of the stationary state).

12. (6 points) State and prove the 6-color theorem for planar graphs. State but do not prove the result used in the proof.

13. (Challenge problem) Let G be a directed graph such that the outdegree of every vertex is $\leq k$. Prove that G is $(2k+1)$-colorable. (A legal coloring of a directed graph is just a legal coloring of the corresponding undirected graph abtained by ignoring orientation.)