Not Proofread

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 20: May 4, 2005

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NOTE: Change in Monday's TA schedule; no change Tuesday and Thursday.
TA SCHEDULE: TA sessions are held in Ryerson-255, Monday 7:30-8:30,
Tuesday and Thursday 5:30–6:30pm.
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Finite Projective Planes

In class H_i we consider random variables $\{(\deg(v) \mid G_[A] = G_i \; for \; j = 1, \ldots, \frac{n}{2}\}$ where $G[A]$ isthe subgraph of G induced on A and $j = 1, \ldots, \frac{n}{2}$ are vertices in A.

Exercise 20.1 *Finish the proof that the number of different degrees in a graph G on n vertices is $\Omega(\sqrt{n})$ almost always.*

Finite projective planes

Defnition: The *incidence geometry* consists of 3 parts. $\mathcal{G} = (P, L, I)$ where P is the set of *points*, L is the set of *lines* and $I \subseteq P \times L$ is the *incidence relation*.
If $(p, \ell) \in I$ then we write $P \longrightarrow \ell$ (p, ℓ are incident).
If $(p, \ell) \notin I$ then we write $P \not\longrightarrow \ell$ (p, ℓ are not incident).
Definition: The *dual* of $\mathcal{G} = (L, P, I)$ is $\mathcal{G}^* = (L, P, I^{-1})$ where $I^{-1} := \{(x, y) : (y, x) \in I\}$.
Definition: A (possibly degenerate) *projective* plane is an incidence geometry such that
(A1) $(\forall p_1, p_2 \in P)(\exists$ *a unique* $\ell \in L)(p_1, p_2$ *are incident on*$\ell)$
(A2) $(\forall \ell_1 \neq \ell_2 \in L)(\exists$ *a unique* $p \in P)(p$ *incident on* $\ell_1, \ell_2)$

Exercise 20.2 *Show that in (A2) uniqueness is redundant.*

Exercise 20.3 *If $\mathcal{G}$ is a possibly degenerate projective plane then so is $\mathcal{G}^*$. (Hint: A1 is A2.)*

(A3) Not all points are on a line, i.e., there exists 4 points such that no 3 are on a line.
Definition: The *degenerate projective plane* is an incidence geometry satisfying A1, A2 and A3.

Exercise 20.4 *If $\mathcal{G} = (P, L, I)$ is a possibly degenerate projective plane the $|P| = |L|$.*

Exercise 20.5 *If $\mathcal{G}$ is a possibly degenerate projective plane and p not incident onℓ ....*

Exercise 20.6 *If $\mathcal{G}$ is a projective plane and $p_1, p_2 \in P$ then $\exists \ell \in L$ such that p_1, p_2 incident on ℓ.*

Corollary 20.7 *In a projective plane all the points have the same degree. Hence all the lines have the same degree by duality.*

Exercise 20.8 *Call this canonical degree (n+1). $n =$ order of $\mathcal{G}$.*