CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 19: May , 2005

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NOTE: Change in Monday's TA schedule; no change Tuesday and Thursday.
TA SCHEDULE: TA sessions are held in Ryerson-255, Monday 7:30-8:30,
Tuesday and Thursday 5:30–6:30pm.
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Martingales

Definition: If X, Y are random variables in the same probability space then the *conditional expectation of* X *conditioned on* Y, is a random variable $Z := E(X|Y) \;:\; \Omega \to \mathbb{R}$ defined as $Z(a) = E(X|Y = Y(a))$.
Some Observations:
(i) If c is a contant random variable, then $E(X|c) = E(X)$.
(ii) The number of different values taken by $E(X|Y) \leq$ the number of different values taken by Y.
(iii) If X and Y are independent then $E(X|Y) = E(X)$.
(iv) $E(X|X) = X$.
(v) $E(X^2|X) = X^2$.
$E(X|X^2)$ has no simple formula. It depends on the proportion of the positive and negative values of X in each part $X^2 = a$.

Definition: The *stochastic process* is a sequence of random variables, $X_0, X_1, X_2, \dots$.
Example of a stochastic process: Gambler's ruin
Suppose a gambler starts with amount X_0. At every gamble, he loses with probability $\frac{1}{2}$ and wins with probability $\frac{1}{2}$. The amount decreases by 1 when he loses and increases by 1 when he wins. X_i is the random variable denoting the amount that the gambler has after i gambles. $X_0, X_1, \dots$ is a stochastic process.
$E(X_1) = X_0$.
$E(X_2) = X_0$. (Why?)
$E(X_2|X_1) = X_1$.
$E(X_{i+1}|X_i) = X_i$.
Definition: The *martingale* is a stochastic process in which $(\forall i)(E(X_{i+1}|X_i) = X_i)$.

Theorem 19.1 (Hoeffding-Azuma: Concentration Inequality) *Suppose* $X_0 = c$, *a constant.* $X_0, X_1, X_2, \dots, X_n$ *is a martingale and* $(\forall i)(|X_{i+1} - X_i| \leq 1)$ *then*

$P(|X_n - X_0| \geq a) \leq 2\mathrm{e}^{\frac{-a^2}{2n}}$.

The Chernoff's bound is a special case of the above theorem. (Why?)

Exercise 19.2 *For $i = 0, 1, \ldots$, let Y_i be independent random variables such that $|Y_i| \leq 1$ and $E(Y_i) = 0$. Let $X_i = Y_1 + \cdots + Y_i$. Prove that $X_0, X_1, dots$ is a martingale.*

Doob-martingale
Let $Y_0 = c$, a constant. Let $Y_0, Y_1, \ldots, Y_n$ be a stochastic process.
Let $X_i = E(Y_n|Y_i)$. So $X_0 = E(Y_n)$. $X_n = E(Y_n|Y_n) = Y_n$. $X_0, X_1, \ldots X_n$ is called Doob-martingale.

Exercise 19.3 $E(X_i) = E(Y_n) = X_0$. $E(X_{i+1}|X_i) = X_i$.

Chromatic number of random graphs Recall that for any graph G, $\chi(G) \geq \frac{n}{\alpha(G)}$.

Theorem 19.4 *For a random graph G on n vertices, almost always $\alpha(G) = \Theta(\log n)$, i.e., $P(\alpha(G) = \Theta(\log n)) \to 1$ as $n \to \infty$.*

Exercise 19.5 $\chi(G) = \Theta(\frac{n}{\log n})$ *almost always.*

Theorem 19.6 $\exists k(n))(\forall g(n) \to infty)(almost\ always\ |\chi(G) - k(n)| < g(n)\sqrt{n})$.

Exercise 19.7 *Modify the proof of Chernoff to prove this.*

Exercise 19.8 *Prove the theorem.*