Not Proofread (5-6-2005)

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 17: May 6, 2005

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NOTE: Change in Monday's TA schedule; no change Tuesday and Thursday.
TA SCHEDULE: TA sessions are held in Ryerson-255, Monday 7:30–8:30 ,
Tuesday and Thursday 5:30–6:30pm.
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The second moment method

Lemma 17.1 (The second moment method) *If $X \geq 0$ is a random variable not identically zero, then $P(X = 0) \leq \frac{var(X)}{E(X)^2}$.*

Recall Chebychev's inequality. $P(|X - E(X)| \geq a) \leq \frac{\mathrm{var}(X)}{a^2}$
Let $a := E(X) > 0$.

$$\begin{aligned} P(X = 0) &\leq P(|X - E(X)| \geq E(X)) \\ &\leq \frac{\mathrm{var}(X)}{a^2} \\ &= \frac{\mathrm{var}(X)}{(E(X))^2}. \end{aligned}$$

$\mathrm{var}(X) = \sum_i \mathrm{var}(X_i) + 2\sum_{i<j} \mathrm{cov}(Y_i, Y_j) < 2\sum_{i\leq j} \mathrm{cov}(Y_i, Y_j)$.
$\mathrm{cov}(S, T) = E(ST) - E(S)E(T)$.
□

Let X be the number of runs of k heads in a sequence of n coin flips.

Theorem 17.2 *1. $n = n(k)$. If $\frac{n(k)}{2^k} \longrightarrow 0$ then $P(X \neq 0) \longrightarrow 0$.*

2. If $\frac{n(k)}{2^k} \longrightarrow \infty$ then $P(X \neq 0) \longrightarrow 1$. ("Almost surely, there exists a run of k heads"). $Y = Y_1 + \cdots + Y_{n-k+1}$. Y_i indicates the event that the $i^{th}, \ldots, (i + k - 1)^{th}$ coin flips come out heads.

Proof:
$E(Y_i) = P(Y_i = 1) = 1/2^k$. Therefore $E(X) = \sum_{i=1}^{n-k+1} E(Y_i) = \frac{n-k+1}{2^k}$. To prove (1.), observe that $P(X \neq 0) = P(X \geq 1) \leq E(X)$. To prove (2.), we use the *second moment method.*

Claim 17.3 *var*$(X) < 5(E(X))$

Proof of Theorem 17.2 using Claim 17.3 :
$\text{var}(X) < cE(X)$.
$P(X = 0) \leq \frac{\text{var}(X)}{E(X)^2} < \frac{c}{E(X)} \longrightarrow 0.$
□
Proof of Claim 17.3:
Suppose $|j - i| = k - \ell$, $\ell \geq 1$ being the "size of the overlap".

$$\begin{aligned} \text{cov}(Y_i Y_j) &= E(Y_i Y_j) - E(Y_i)E(Y_j) \\ &< E(Y_i Y_j) \\ &< 2^{-(2k-\ell)} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i \leq j} \text{cov}(Y_i, Y_j) &< \sum_{\ell=0}^{k} \frac{n-k+1}{2^{2k-\ell}} \\ &= \frac{n-k+1}{2^{2k}} \sum_{\ell=0}^{k} 2^\ell \\ &< \frac{n-k+1}{2^{2k}} \sum_{\ell=0}^{k} 2^\ell \\ &< 2\frac{n-k+1}{2^k} \\ &= 2E(X) \end{aligned}$$

Therefore, $\text{var}(X) \leq 2\sum_{i \leq j} \text{cov}(Y_i, Y_j) < 4E(X)$. □

Theorem 17.4 *For almost all graphs on n vertices, $\alpha(G) \sim 2\log_2(n)$.*

We already know : $\lesssim 2\log_2 n$. We need: $\gtrsim 2\log_2 n$.
Method: 2^{nd}-Moment method. (Hint: X = the number of anticliques of size k.)

Definition 17.5 *$\tau(\mathcal{H})$ is the covering number of a k-uniform hypergraph with m edges and n vertices.*
i. e. $\tau(\mathcal{H}) = \min |T| : T$ *such that* $(\forall i)(T \cap A_i \neq \emptyset)$.

Claim 17.6 $\tau(\mathcal{H}) < 1 + n/k \ln(m)$.

Claim 17.7 *If T is a random subset of size t, $t \geq n/k \ln m$, then with probability greater than 0, T is a cover.* $P(A_i$ *is not hit by* $T) = \frac{\binom{n-k}{t}}{\binom{n}{t}}$.

Proof:

$$P(A_i \text{ is not hit by } T) = \frac{\binom{n-k}{t}}{\binom{n}{t}}.$$

$$P((\exists i) A_i \text{ is not hit by } T) < m \frac{\binom{n-k}{t}}{\binom{n}{t}} \leq 1.$$

□

Exercise 17.8

$$\begin{aligned} \frac{\binom{n-k}{t}}{\binom{n}{t}} &\leq (\frac{n-k}{n})^t \\ &= (1 - k/n)^t < e^{-kt/n} \\ &\leq e^{-\ln m} = 1/m \end{aligned}$$