Re-revised notes 4-22-2005 10pm

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 10: April 20, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
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IMPORTANT: Take-home test Friday, April 29, due Monday, May 2, **before** class.

Perfect Graphs

Shannon capacity of a graph G is: $\Theta(G) := \lim_{k\to\infty} \left(\alpha(G^k)\right)^{1/k}$.

Exercise 10.1 *Show that $\alpha(G) \le \chi(\overline{G})$. ($\overline{G}$ is the complement of G.)*

Exercise 10.2 *Show that $\chi(\overline{G \cdot H}) \le \chi(\overline{G})\chi(\overline{H})$.*

Exercise 10.3 *Show that $\Theta(G) \le \chi(\overline{G})$.*

So, $\alpha(G) \le \Theta(G) \le \chi(\overline{G})$.

Definition: G is *perfect* if for all induced sugraphs H of G, $\alpha(\overline{H}) = \chi(H)$, i.e., the chromatic number is equal to the clique number.

Theorem 10.4 (Lovász) *G is perfect iff $\overline{G}$ is perfect.*
(This was open under the name "weak perfect graph conjecture.")

Corollary 10.5 *If G is perfect then $\Theta(G) = \alpha(G) = \chi(\overline{G})$.*

Exercise 10.6 *(a) K_n is perfect. (b) All bipartite graphs are perfect.*

Exercise 10.7 *Prove: If G is bipartite then $\overline{G}$ is perfect. Do not use Lovász's Theorem (Theorem 10.4).*

The smallest imperfect (not perfect) graph is C_5 : $\alpha(\overline{C_5}) = 2$, $\chi(C_5) = 3$.
For $k \geq 2$, C_{2k+1} imperfect.

Definition: A graph is *minimally imperfect* if it is imperfect but deleting any vertex leaves a perfect graph.

Exercise 10.8 *For $k \geq 2$, C_{2k+1} and its complement are minimally imperfect.*

The Perfect Graph Conjecture (Berge): These are the only minimally imperfect graphs. This was proved recently in a monumental paper:

Theorem 10.9 (Perfect Graph Theorem) (Maria Chudnovsky, Neil Robertson, Paul Seymour, Robin Thomas, 2005) *The odd cycles of length ≥ 5 and their complements are all the minimally imperfect graphs.*

Definition: A *partially ordered set* (poset) $\mathcal{P} = (S, R)$ is a set S with a relation $R \subseteq S \times S$ such that R is
(a) reflexive ($(x, x) \in R$)
(b) symmetric ($(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$)
(c) transitive ($(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$).
R is usually denoted by "$\leq$," so instead of "$(x, y) \in R$" we write "$x \leq y$."
Examples: 1) Family of sets with respect to inclusion.
2) Positive integers with respect to divisibility.
Definitions: (i) In a poset $\mathcal{P} = (S, \leq)$, a and b are *comparable* if $a \leq b$ or $b \leq a$.
(ii) The *comparability graph* of $\mathcal{P} = (S, \leq)$ is a graph $G = (S, E)$, where $E = \{$comparable pairs of distinct elements of $S\}$.
(iii) A clique in the comparability graph of a poset is a *chain* in the poset: $a_1 < a_2 < \cdots < a_k$.
Example of a chain among integers with respect to divisibility: $2|6|42|210$.
(iv) An independent set in the comparability graph is called an *antichain*, e.g. $\{10, 12, 35\}$

Observation. If G is the comparability graph of a poset $\mathcal{P} = (S, \leq)$ then the chromatic number of G is the minimum number of colors to color S such that the vertices of each color form an antichain.

Exercise 10.10 *Show that $\chi(G) =$ size of a maximum chain.*
(Hint: To prove $\leq$, use induction on the length of maximum chain.)

Observation: An induced subgraph of a comparability graph is a comparability graph.

Corollary 10.11 *Comparability graphs are perfect.*

Using Lovász Theorem (Theorem 10.4), we have:

Corollary 10.12 *Incomparability graphs are perfect.*

This translates to Dilworth's celebrated theorem:

Corollary 10.13 (Dilworth, 1947) *The size the largest antichain in a poset = minimum number of chains into which the poset can be partitioned.*

Exercise 10.14 *Prove: The size the largest antichain in a poset $\leq$ minimum number of chains into which the poset can be partitioned. (Hint: PHP.)*
(This is the trivial direction of Dilworth's theorem.)

Definition: The *power-set* of a set S := set of all subsets of S.
This is a poset under inclusion. An antichain of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ can be found in the power-set of S if $|S| = n$. (Take all subsets of size $\lfloor \frac{n}{2} \rfloor$.)

Theorem 10.15 (Sperner's Theorem) *If $A_1, \ldots, A_m \subseteq [n]$, are pairwise incomparable, then $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Exercise 10.16 * *Prove Sperner's Theorem by dividing the power-set into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains.*

Theorem 10.17 (LYM inequality) *If $A_1, \ldots, A_m \subseteq [n]$, are pairwise incomparable, then $\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1$.*

An antichain of sets is also called a "Sperner family."

Exercise 10.18 *Prove: The LYM inequality implies the Sperner's Theorem.*

Exercise 10.19 *Let $r_1, r_2, \ldots, r_n > 0$ real numbers, $b > 0$. Show that $P(\sum_{i=1}^{n} a_i r_i = b) \leq \frac{c}{\sqrt{n}}$ where the coefficients a_i are decided by coin tosses: we set $a_i = 1$ if the i-th coin comes up Heads and $a_i = 0$ if the i-th coin comes up Tails.*

Exercise 10.20 (Ramsey number $R(3,4)$) *(a)* $9 \longrightarrow (3,4)$. *(b)* $8 \not\longrightarrow (3,4)$.