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CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 1: March 28, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Tuesday and Thursday 5:30–6:30pm.
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Ramsey Theory for graphs

Notation 1.1 (Erdös-Rado arrow notation) *Let $(\ell_1, \ldots, \ell_k)$ be a k-tuple of natural numbers. If N has the property that any edge-coloring of the complete graph K_N using the colors $c_1, \ldots, c_k$ contains for some i, a clique of size ℓ_i monochromatic with color c_i, then we write*

$$N \longrightarrow (\ell_1, \ldots, \ell_k).$$

If N does not have this property, we write

$$N \not\longrightarrow (\ell_1, \ldots, \ell_k).$$

Exercise 1.2 (a) $5 \not\longrightarrow (3,3)$; (b) $6 \longrightarrow (3,3)$; (*c*) $17 \longrightarrow (3,3,3)$.

Exercise 1.3 (Erdös-Szekeres) $\binom{k+\ell}{k} \longrightarrow (k+1, \ell+1)$.

Theorem 1.4 $\infty \longrightarrow (\infty, \infty)$.

Proof:
Let the colors be *red* and *blue*. Label the vertices of the graph $1, 2, \ldots$ (the number of vertices is assumed to be countable). If the vertex 1 has infinitely many red edges incident upon it, color 1 red and discard all vertices that are connected to 1 via a blue edge, otherwise color it blue and discard all vertices that are connected to 1 via a red edge to get $S_1 \subseteq \mathbb{N}$. Suppose now, that we have colored a subsequence $1 = s_1 < s_2 < s_3 < \cdots < s_k$ of the vertices, and have discarded all vertices not in the set S_k. We pick the first vertex of S_k greater than s_k and call it s_{k+1}. If the vertex s_{k+1} has infinitely many red edges connected to nodes in S_k, color s_{k+1} red and discard all vertices that are connected to s_{k+1} via a blue edge with

label greater than s_{k+1}, otherwise color it blue and discard all vertices beyond s_{k+1} that are connected to it via a red edge . Let $S = \cap_k S_k$, and let S inherit colors from the S_k. The red vertices in S induce a red clique and the blue vertices induce a blue clique. One of these is infinite since S is. □

Using the same argument, we have

Theorem 1.5 (Infinite Ramsey theorem for graphs) $(\forall k)\infty \longrightarrow \underbrace{(\infty, \ldots, \infty)}_{k}$.

A *rooted graph* is a graph with a vertex designated as the *root.*

Lemma 1.6 (König Path Lemma) *Let G be a directed rooted graph. Assume that*

1. *The out-degree of every vertex is finite.*
2. *All vertices are accessible from the root.*
3. *G is infinite.*

Then there exists an infinite path from the root.

Proof: Without loss of generality we may assume the graph to be a rooted tree. (Otherwise we replace the graph by its BFS tree.) The root has infinitely many descendants, but finitely many children. Therefore one of its children has infinitely many descendants. Repeat the argument with such a child. □

Theorem 1.7 (Finite Ramsey Theorem for graphs) $(\forall \ell_1, \ldots, \ell_k)(\exists R)(R \longrightarrow (\ell_1, \ldots, \ell_k))$.

Proof:

Proof by contradiction. Suppose statement is false. Then $(\exists \ell_1, \ldots, \ell_k)(\forall R)(R \not\longrightarrow (\ell_1, \ldots, \ell_k))$. Let us fix a set of k colors. Consider the complete graph K_n with vertex set $\{1, \ldots, n\}$. We say that an edge-coloring of K_n by the given k colors is a "counterexample" if there is no clique of size ℓ_i in the i-th color. By our assumption, there is a counterexample for every n.

Define a directed graph G of which the vertices are all the finite counterexamples. Let the 1-vertex graph (no colors) be the root.

Let X be a countexample with $n+1$ vertices and Y a counterexample with n vertices. We put a $Y \to X$ edge in G if Y is the restriction (induced edge-colored subgraph) of X to the vertex set $\{1, \ldots, n\}$. This completes the definition of the directed graph G. (In fact, G is a directed tree, directed away from the root.) Note that all outdegrees are finite and all vertices are reachable from the root.

By the König Path Lemma, there is an infinite path from the root. The union of the corresponding chain of k-colored finite complete graphs is an infinite complete graph (because every pair of positive integers gets only one color assigned, the same color in all colorings along the path); and this union has no clique of size ℓ_i in color i. The reason is that if the union did contain such a clique, say on vertices $v_1, \ldots, v_{\ell_i}$, then than this clique would already be present in the coloring found along the path of K_n where $n = \max_j v_j$.

But the existence of such a k-coloring contradicts the Infinite Ramsey Theorem for graphs. □

Exercise 1.8 *Prove: 3-Colorability is a finitary property of graphs. In other words a graph is 3-colorable if and only if every finite subgraph of it is.*

For this exercise, assume that the graph is countable; use the König Path Lemma. For the full strength of the result, do not assume the graph is countable and use Zorn's Lemma.