Proofread (5-4-2005)

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 15: May 2, 2005

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NOTE: Change in Monday's TA schedule; no change Tuesday and Thursday.
TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
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Variance and Covariance

Definition 15.1 *Let $X_1, \ldots, X_k : \Omega \longrightarrow \mathbb{R}$ be random variables on the same probability space (Ω, P). $(X_1, \ldots, X_k)$ are independent if $\forall x_1, \ldots, x_k \in \mathbb{R}$,*

$$P[X_1 = x_1, \ldots, X_k = x_k] = \prod_{i=1}^{k} P[X_i = x_i].$$

Definition 15.2 *Events $A_1, \ldots, A_k$ are independent if their indicator variables are independent.*

Exercise 15.3 *Prove that this definition is equivalent to the definition in the handout.*

Theorem 15.4 *If $X_1, \ldots, X_k$ are independent random variables then $E(\prod_{i=1}^k X_i) = \prod_{i=1}^k E(X_i)$.*

Exercise 15.5 *(a) Construct random variables X, Y such that $E(XY) = E(X)E(Y)$, but X and Y are not independent. Do this with $|\Omega| = 3$.*
(b) If $|\Omega| = 2$, then $E(XY) = E(X)E(Y) \implies X, Y$ independent.

Definition 15.6 *The covariance of X and Y is $cov(X, Y) := E(XY) - E(X)E(Y)$.*

Definition 15.7 *X, Y are positively correlated if $cov(X, Y) > 0$; negatively correlated if $cov(X, Y) < 0$, and uncorrelated if $cov(X, Y) = 0$.*

Definition 15.8 *The variance of X, $var(X) = E((X - E(X))^2)$.*

Theorem 15.9 (Markov's inequality) *If* $X \geq 0$ *(non-negative random variable), then* $(\forall a > 0) P(X \geq a) \leq E(X)/a$.

Definition 15.10 *Standard deviation* $\sigma(X) := \sqrt{var(X)}$.

Theorem 15.11 (Chebychev's inequality)

$$\begin{aligned} P(|X - E(X)| \geq a) &\leq \frac{var(X)}{a^2} \\ &= (\frac{\sigma(X)}{a})^2 \end{aligned}$$

Proof: $Y := (X - E(X))^2 \geq 0$. $E(Y) = \text{var}(X)$. $P(|X - E(X)| \geq a) = P(Y \geq a^2) \leq E(Y)/a^2$. □

Definition 15.12 Bernoulli trials *have two outcomes - "H" with probability* p *and "T" with probability* $1 - p$. *The probability that out of* n *independent trials, there are* k *heads is* $\binom{n}{k} p^k (1-p)^{n-k}$. *This distribution is called the* binomial distribution.

Theorem 15.13 *If* X *has the binomial distribution,* $E[X] = np$.

Proof: $X = Y_1 + \cdots + Y_n$ $Y_i = 1$ if the i^{th} coin comes up Heads. $E(Y_i) = p$. Therefore $E(X) = \sum E(Y_i) = np$. □

Let Y be a bernoulli trial.

$$\begin{aligned} \text{var}(Y) &= E(Y^2) - (E(Y))^2 \\ &= p - p^2 \end{aligned}$$

Suppose $X = Y_1 + \cdots + Y_n$,

$$\begin{aligned} \text{var}(X) &= E((\sum Y_i)^2) - (E(\sum Y_i))^2 \\ &= E(\sum_{i=1}^{n} \sum_{j=1}^{n} Y_i Y_j) - (\sum E(Y_i))^2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} E(Y_i Y_j) - E(Y_i) E(Y_j). \end{aligned}$$

Theorem 15.14

$$\begin{aligned} var(X) &= \sum_{i=1}^{n} \sum_{j=1}^{n} cov(Y_i, Y_j) \\ &= \sum_{i=1}^{n} var(Y_i) + 2 \sum_{i<j} cov(Y_i, Y_j) \end{aligned}$$

Corollary 15.15 *If $Y_1, \ldots, Y_n$ are pairwise independent, then var$(\sum_{i=1}^n Y_i) = \sum_{i=1}^n$ var(Y_i).*

If X_n has the binomial distribution with parameters (n, p), $\text{var}(X) = np(1-p)$.

Theorem 15.16 (Weak Law of Large Numbers) $(\forall \epsilon > 0) \lim_{n \longrightarrow \infty} P(|X_n/n - p| > \epsilon) = 0$

Proof:

$P(|X_n - np| > n\epsilon) < \frac{\text{var}(X_n)}{(n\epsilon)^2} = \frac{np(1-p)}{n^2\epsilon^2} = o(1)$

Experiment: n unbiased coin flips.

Let X be the number of runs of k heads. $n = n(k)$. If $\frac{n(k)}{2^k} \not\longrightarrow \infty$, $P(\exists \text{run of k heads}) \longrightarrow 0$, since $P(\exists \text{run of k heads}) < \frac{n-k+1}{2^k}$. Now assume $\frac{n(k)}{2^k} \longrightarrow \infty$.

Claim 15.17 *Almost surely, there exists a run of k heads.*

i. e. $P(X \neq 0) \longrightarrow 1$.
i. e. $P(X = 0) \longrightarrow 0$.
$P(X = 0) \leq P(|X - E(X)| = E(X)) \leq \frac{\text{var}(X)}{E(X)^2}$ □

Exercise 15.18 *Prove if $\frac{n(k)}{2^k} \longrightarrow \infty$ then var$(X) = o(E(X)^2)$.*