Proofread notes 5-4-2005

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 14: April 29, 2005

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NOTE: Change in Monday's TA schedule; No change Tuesday and Thursday
TA SCHEDULE: TA sessions are held in Ryerson-255, Monday 7:30-8:30pm,
Tuesday and Thursday 5:30–6:30pm.
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Sperner's theorem

Theorem 14.1 (Sperner's theorem) *If $A_1, \ldots, A_m \subseteq [n]$ is a Sperner family (i. e.antichain) then $m \leq \binom{n}{\lfloor n/2 \rfloor}$.*

Exercise 14.2 *Every uniform set system is Sperner.*

Lemma 14.3 (BLYM inequality) *If $A_1, \ldots, A_m$ is a Sperner family, then*

$$\sum_{i=1}^{m} \binom{n}{A_i} \leq 1.$$

Note: LYM : Lubell-Yamamoto-Meshalkin. The inequality is usually referred to as "LYM inequality" even though Béla Bollobás, then an undergraduate, proved it first and in a more general form.
We prove that BLYM inequality implies Sperner's theorem. **Proof:** Suppose $A_1, \ldots, A_m$ is a Sperner family.

$$1 \leq \sum_{i=1}^{m} \binom{n}{|A_i|}^{-1} \leq \sum_{i=1}^{m} \binom{n}{\lfloor n/2 \rfloor}^{-1} = \frac{m}{\binom{n}{\lfloor n/2 \rfloor}}. \qquad \square$$

We now prove the BLYM inequality.
Let 2^A be the set of all subsets of A. $|2^A| = 2^{|A|}$ and $(2^A, \subseteq)$ is a poset. Let $|A| = n$. Maximal chains in 2^A have length $n+1$. The number of maximal chains is $n!$.
Let $\mathcal{C}$ be a random maximal chain in $2^{[n]}$.
Let $X := |\{i \mid A_i \in \mathcal{C}\}|$
(This is a random variable.) The Sperner property implies $X \leq 1$. (Why?). Now $X = \sum Y_i$;

$Y_i = 1$ if $A_i \in \mathcal{C}$, and 0 otherwise. So Y_i is the indicator of the event "$A_i \in \mathcal{C}$."
$E(X) = \sum_{i=1}^{m} E(Y_i)$. Summarizing,

$$1 \geq E(X) = \sum_{i=1}^{m} \binom{n}{|A_i|}^{-1}.$$

$E[Y_i] = P(A_i \in \mathcal{C}) = P(A_i \text{ is a prefix in a random ordering of } [n]) = \binom{n}{|A_i|}^{-1}$ (by symmetry, since A_i could be any of the $\binom{n}{|A_i|}$ subsets of size $|A_i|$ under the random ordering). □

Let $m(k)$ denote minimum number of edges in a k-uniform hypergraph which is not 2-colorable. So $m(2) = 3$.

Theorem 14.4 (Erdős) $2^{k-1} < m(k) < ck^2 2^{k-1}$.

For the hypergraph to be 2-colorable, there must exist a partition $R \dot{\cup} B$ of $\bigcup A_i$ such that $(\forall i)(A_i \not\subseteq B \text{ and } A_i \not\subseteq R)$. We shall only prove the lower bound $2^{k-1} < m(k)$ here.
Comment on upper bound: proof by probabilistic method; no constructive proof known. But we can prove constructively that $m(k) < 4^k$.
For the hypergraph to be 2-colorable, there must exist a partition $R \dot{\cup} B$ of $\bigcup A_i$ such that $(\forall i)(A_i \not\subseteq B \text{ and } A_i \not\subseteq R)$.

K_{2k-1}^k is not 2-colorable - by the Pigeon-hole Principle, some k vertices must have the same color. Thus

$$m(k) \leq \binom{2k-1}{k} < 2^{2k-1} < 4^k.$$

Proof of the lower bound.
If $m \leq 2^{k-1}$, then the hypergraph is always 2-colorable. Color the vertices Red and Blue at random.

Claim 14.5

$$P(\textit{Coloring illegal}) < 1.$$

Proof: $P(A_i \text{ monochromatic}) = 2^{1-k}$.
$P(\exists i \text{ such that } A_i \text{ monochromatic }) < m2^{1-k} \leq 1$ (Union Bound)
The inequality is strict because the events "A_i monochromatic" overlap, for example, when all vertices have the same color. □

Large graphs without 4-cycles

Recall Thm 4.7 (Kővári, Sós, Turán):
$\mathrm{ex}(n, C_4) = \mathrm{O}(n^{3/2})$.
Here is an example to show that this bound is tight. We need to construct graphs on n vertices and $\Omega(n^{3/2})$ edges, without 4-cycles. Consider the two dimensional plane over F_p of

integers (mod p). The points are $P = F_p \times F_p$; lines are sets of points satisfying a linear equation (mod p):

$$ax + by + c \equiv 0 (\text{mod } p),$$

where not both a and b are $\equiv 0(\text{mod } p)$. Consider the "Levy Graph" of the plane whose vertex set is $P \cup L$. An edge joins $p \in P$ and $\ell \in L$ if and only if p is a point lying on line ℓ. Therefore, in the Levy graph, the number of vertices $n = p(2p + 1)$ $2p^2$ (why?), and the number of edges $m = p^2(p + 1)$ $2p^3$ (why?). Thus $m = \Theta(n^{3/2})$. □