CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 12: April 25, 2005

Instructor: László Babai *Scribe: Raghav Kulkarni*

TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
INSTRUCTOR'S EMAIL: laci@cs.uchicago.edu
TA's EMAIL: hari@cs.uchicago.edu, raghav@cs.uchicago.edu
IMPORTANT: Take-home test Friday, April 29, due Monday, May 2, **before** class.
READING: Review the "Finite Probabilty Spaces" handout (all of it except the last section, 7.5 ("Chernoff's bound").

Finite Probability Spaces

Terminology: Let $\Omega \neq \emptyset$ (finite) be the *sample space*: set of possible outcomes of an experiment.
Examples: (i) Picking a poker hand (picking 5 out of 52 cards) is an experiment. The size of the sample space for this experiment is $\binom{52}{5.}$
(ii) Flipping n coins is an experiment. The sample space for this experiment has 2^n elements.
(iii) Choosing a random graph by flipping a coin for every pair of vertices is an experiment. The size of the sample space here is $2^{\binom{n}{2}}$.
The elements of Ω are called *elementary events*.
Definition: The *probability distribution* on a sample space Ω is a function
$P : \Omega \longrightarrow \mathbb{R}$ such that
(a) $(\forall x \in \Omega)(P(x) > 0)$
(b) $\sum_{x\in\Omega} P(x) = 1$.
To every $x \in \Omega$, the probability distribution assigns the value $P(x)$: *the probability of elementary event* x.
Definition: The *probability space* (Ω, P) is a sample space Ω with a probability distribution on it.
Definition: P is the *uniform distribution* if $(\forall x \in \Omega)(P(x) = \frac{1}{|\Omega|})$.
Definition: An *event* is a subset $A \subseteq \Omega$.
Definition: The *probability of an event* A is the sum of the probabilities of the elementary events in it. $P(A) = \sum_{x\in A} P(x)$.

Exercise 12.1 *(a) If* $A_1, \ldots, A_k$ *are disjoint events, i. e.,* $(\forall i \neq j)(A_i \cap A_j = \emptyset)$, *then* $P(\bigcup_{i=1}^k A_i) = \sum_{i=1}^k P(A_i)$.
(b) (The **Union Bound**) $(\forall A_1, \ldots, A_k \subseteq \Omega)(P(\bigcup_{i=1}^k A_i) \leq \sum_{i=1}^k P(A_i))$.

Note that The Union Bound was used to prove $n \not\longrightarrow (1 + 2\log_2 n, 1 + 2\log_2 n)$. (Lecture 3, Theorem 3.3).
Convention: If P is not mentioned in the specification of a probability space then P is assumed to be the uniform distribution.

Exercise 12.2 *If P is uniform and $A \subseteq \Omega$ then $P(A) = \frac{|A|}{|\Omega|}$.*
(This is called "Naive Probability.")

Definition: A *random variable* on a probability space (Ω, P) is a function $X : \Omega \longrightarrow \mathbb{R}$.
Notice that the probability distribution is itself a random variable.
Examples of random variables: (i) The number of heads in a sequence of n coin flips.
(ii) The number of spades in a poker hand.
(iii) The size of the largest clique in a random graph.
(iv) The chromatic number of a random graph.
Definition: The *expected value* of a random variable X: $E(X) := \sum_{x \in \Omega} X(x)P(x)$.
In other words, the expected value of a random variable X is the weighted average of the values of X.

Exercise 12.3 *Prove:* $E(X) = \sum_{y \in \mathbb{R}} yP(X = y)$.

Definition: The *indicator variable* of an event $A \subseteq \Omega$ is a random variable defined as follows: $\theta_A(x) = 1$ if $x \in A$; $\theta_A(x) = 0$ if $x \notin A$.

Exercise 12.4 *The expected value of an indicator variable is the probability of the event indicated:*

$$E(\Theta_A) = P(A).$$

(Hint: use Exercise 12.3. Observe that the event $\{\Theta_A = 1\} = A$.) (Remember: events have probablilities, random variables have expectations, not vice versa.)

Exercise 12.5 (Linearity of Expectation) *Show that* $E(\sum_{i=1}^{k} \lambda_i X_i) = \sum_{i=1}^{k} \lambda_i E(X_i)$, *where* $X_1, \ldots, X_k$ *are random variables on the probability space* (Ω, P).

A special case of the above exercise is (a) $E(\sum_{i=1}^{k} X_i) = \sum_{i=1}^{k} E(X_i)$
(b) $E(cX) = cE(X)$ for any random variable X.
Definition: The *constant* random variable c is a random variable which is a constant function. $E(c) = c$.

Exercise 12.6 *Show that* $\min X \leq E(X) \leq \max X$ *for any random variable* X.

Example illustrating linearity of expectation: Let Ω be the sample space for the experiment of picking a poker hand. P is the uniform distribution on Ω. Let X be a random variable on probability space Ω defined as follows: $X =$ the number of *kings* in the poker hand. What is $E(X)$?
Let us denote the kings of different suits by $K\heartsuit, K\spadesuit, K\diamondsuit, K\clubsuit$.
Let θ_1 be the indicator variable for the event "the hand includes $K\heartsuit$."
Let θ_2 be the indicator variable for the event "the hand includes $K\spadesuit$."
Let θ_3 be the indicator variable for the event "the hand includes $K\diamondsuit$."
Let θ_4 be the indicator variable for the event "the hand includes $K\clubsuit$."
Then, $X = \theta_1 + \theta_2 + \theta_3 + \theta_4$.
By the linearity of expectation, $E(X) = E(\theta_1) + E(\theta_2) + E(\theta_3) + E(\theta_4)$.
$E(\theta_1) = P(\theta_1 = 1) = P(K\heartsuit \textit{ is in hand}) = \frac{5}{52}$. (Why? Use Exercise 12.1 (a).) Similarly, $E(\theta_2) = \frac{5}{52}$. $E(\theta_3) = \frac{5}{52}$. $E(\theta_4) = \frac{5}{52}$. Hence, $E(X) = E(\theta_1) + E(\theta_2) + E(\theta_3) + E(\theta_4) = 4\frac{5}{52} = \frac{5}{13}$.
REVIEW: ALL BUT CHAPTER 7.5 of PROBABILITY HANDOUTS, INCLUDING MARKOV AND CHEBYSHEV'S INEQUALITY.
Consider a coin flip sequence of lenght n. A run of heads of length k is a subsequence of k consecutive heads in the sequence.

Exercise 12.7 *Let $A(n,k)$ be the event that there is a run of k consecutive heads in the coin flip sequence of lenght n. Let $X(n,k) :=$ the number of runs of k heads among n coin flips. Let $n = n(k)$ be a function of k.*
(0) Compute $E(X(n,k))$.
(1) Prove: if $\frac{n(k)}{2^k} \to 0$ then $P(A(n(k),k)) \to 0$ as $k \to \infty$. (Hint: Markov's inequality.)
(2) Prove: if $\frac{n(k)}{k2^k} \to \infty$ then $P(A(n(k),k)) \to 1$.
*(3) * Prove: if $\frac{n(k)}{2^k} \to \infty$ then $P(A(n(k),k)) \to 1$. (Hint: Extimate the variance of $X(n,k)$ and then use the Chebyshev's inequality.)*