Proofread notes 4-24-2005

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 11: April 22, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
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IMPORTANT: Take-home test Friday, April 29, due Monday, May 2, **before** class.

Partially ordered sets, Zorn's lemma

Let $\mathcal{P} = (A, \leq)$ be poset. (The number of elements in A may be infinite.)
Definitions:
(i) $B \subseteq A$ is *bounded* if $(\exists a \in A)(\forall b \in B)(b \leq a)$.
(ii) $C \subseteq A$ is a *chain* if $(\forall x, y \in C)(x \leq y \text{ or } y \leq x)$.
(iii) $a \in A$ is *maximal* if $(\nexists x \in A)(a < x)$; here $a < x$ means $a \leq x$ *and* $a \neq x$.
(iv) $a \in A$ is *greatest* if $(\forall x \in A)(x \leq a)$.

Observe that a greatest element is necessarily maximal but not conversely.

Exercise 11.1 *Let $\mathcal{P} = (A, \leq)$ be a poset. True or false: (a) the empty set is a chain. (b) The empty set is bounded in $\mathcal{P}$.*

Exercise 11.2 *(a) Show that every nonempty finite poset has a maximal element.*
(b) Show that a finite poset has a greatest element iff it has a unique *maximal element.*

Exercise 11.3 $\mathbb{N} = \{0, 1, \dots\}$ *(a) with respect to the natural ordering, has no maximal element; (b) with respect to divisibility, has a maximal elements. It also has a minimal element (define!).*

In $(\mathbb{N}, \leq)$, 0 is the greatest and 1 is the smallest element.

Zorn's Lemma: *In a poset, if every chain is bounded then $\exists$ a maximal element.*
Axiom of choice: *If $\{A_i\}_{i \in I}$ is a family of non-empty sets then $\exists$ a function $f : I \longrightarrow \bigcup_{i \in I} A_i$ such that $(\forall i \in I)(f(i) \in A_i)$.*
Zorn's lemma is equivalent to the Axiom of Choice, given the rest of the axioms of set theory.

Theorem 11.4 *Let k be a positive integer. Then k-colorability of graphs is a finitary property, i. e., G is k-colorable iff every finite subgraph of G is k-colorable.*

Proof: $\Rightarrow$ is obvious.
$\Leftarrow$ *Definition:* G is *finitely k-colorable* if every finite subgraph of G is k-colorable.
Let $G = (V, E)$. Let $\mathcal{G}$ be the set of all finitely k-colorable spanning supergraphs of G. (Recall: a *spanning* supergraph of G is a supergraph which has the same set of vertices as G; so a spanning supergraph is specified by specifying a superset of the edges of G.)
$\mathcal{G} = \{H = (V, F) \mid F \subseteq E \textit{ and } H \textit{ is finitely } k\textit{-colorable}\}$. Note that $\mathcal{G}$ is *non-empty.*

Exercise 11.5 *Prove: among all supergraphs of G on the vertex set V which are finitely k-colorable, there is a maximal one. (Hint: Zorn's lemma.)*

Exercise 11.6 *Let H be a maximal k-colorable finite graph, i. e., adding any edge to H destroys the k-colorability. Prove: H is complete k-partite, i. e., the complement of the disjoint union of complete graphs.*

(Note: A finite graphs is k-partite iff it is k-colorable.)

Now back to the proof of Theorem 11.4. Let G_0 be a maximal finitely k-colorable supergraph of G on the same vertex set V. It suffices to prove: **Claim:** G_0 is complete k-partite.
Subclaim: The relation, $a \sim b$: "$a = b$ or a is not adjacent to b in G_0," is an equivalence relation. (i. e., $\overline{G_0}$ is the disjoint union of the complete graphs.)

Proof of Subclaim: Consider three vertices a, b, c in G_0. Suppose $\{a, c\}$ is an edge but $\{a, b\}$ and $\{b, c\}$ are not. Since G_0 is maximal finitely k-colorable, adding the edge $\{b, c\}$ to G_0 makes some finite subgraph H_1 of it not k-colorable. Similarly, adding $\{a, b\}$ makes some finite subgraph H_2 not k-colorable. Consider $H = H_1 \cup H_2$. Every k coloring of H must assign the same color to a and b (since H_1 plus $\{a, b\}$ is not k-colorable) and also it must assign the same color to b and c (since H_2 plus $\{b, c\}$ is not k-colorable). Therefore it will assign the same color to a and c, a contradiction, because a and c are adjacent and every finite subgraph of G_0 is k-colorable. □

Therefore, $\overline{G}_0$ is disjoint union of complete graphs. (Why?)

Therefore, G_0 is a complete ℓ-partite for some (finite or infinite ℓ). So G_0 contains K_ℓ; therefore $\ell \leq k$. (Why?) □

Corollary 11.7 *Borsuk's Theorem implies* $(\forall k, \ell)(\exists$ *a finite graph G with* odd-girth*$(G) \geq \ell$ and G is not k-colorable*$)$.

Exercise 11.8 *Prove the same using Kneser's Conjecture (Lovász's Theorem).*