Revised notes 4-19-2005

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 9: April 18, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
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Shannon capacity and chromatic number

Exercise 9.1 (Infection Puzzle) *Suppose you have a $n \times n$ chessboard. If you put a drop on a cell, it gets infected. The infection can spread in the following way. If two adjacent cells are infected then the cell itself gets infected (Adjacency means sharing a side.) Prove that the minimum number of cells needed to be infected initially so that the infection spreads to the entire board is n.*
*(*This is a difficult problem but it has a 3-line "AH-HA" solution.*)*

Exercise 9.2 *Show that for any graph G, $\alpha(G)\chi(G) \geq n$. (α is the independence number, χ is the chromatic number.)*

This can be used to prove a lower bound for χ by finding an upper bound on α.

Exercise 9.3 *Find graphs with n vertices such that $\alpha(G_n)\chi(G_n) = \Omega(n^2)$.*

Definition: The *strong product* of graphs $G(V, E)$ and $H(W, F)$ is a graph $G \cdot H$ defined as follows: $V(G \cdot H) := V \times W$; we let $(g_1, h_1) \cong (g_2, h_2) \Leftrightarrow g_1 \cong g_2$ and $h_1 \cong h_2$, where $\cong$ means "adjacent or equal to".
$G^k := G \cdot \dots \cdot G$ (k times).

Exercise 9.4 $\alpha(G \cdot H) \geq \alpha(G)\alpha(H)$.

Corollary 9.5 $\alpha(G^{k+\ell}) \geq \alpha(G^k)\alpha(G^\ell)$. *(Hint: Strong product is associative.)*

Corollary 9.6 $\lim_{k\to\infty} (\alpha(G^k))^{1/k}$ *exists. (Hint: Ex. 9.9 below.)*

Observe that $\alpha(G) \le n$. Therefore, $\alpha(G^k) \le |V(G^k)| = n^k$, so, $(\alpha(G^k))^{1/k} \le n$.

Corollary 9.7 $\Theta(G) \le n$.

Exercise 9.8 *(Supermultiplicative sequence.) If* $a_k > 0$, *for all* k *and* $a_{k+\ell} \ge a_k a_\ell$ *then (a)* $L := \lim_{k\to\infty} (a_k)^{1/k}$ *exists.*
(b) $(\forall m)(L \ge a_m^{1/m})$.

Exercise 9.9 *Show that* $\Theta(C_5) \ge \sqrt{5}$. *(Hint: Use* 5×5 *toroidal chessboard. Place the kings in* "knight's straight line path" *(mod 5) to show that* $\alpha(\text{King-graph}) = 5$. *Then use Ex. 9.9 (b).)*

Exercise 9.10 6 *kings cannot be accommodated on a* 5×5 *toroidal chessboard without threatening each other.*

Exercise 9.11 *Show that* $\Theta(C_5) \le \frac{5}{2}$.

Theorem 9.12 (Lovász) $\Theta(C_5) = \sqrt{5}$.

Definition: A graph G is *self-complementary* if it is isomorphic to its complement. (i.e., $G \cong \overline{G}$).

Exercise 9.13 *(a)* C_5 *is self-complementary.*
(b) If G *is self-complementary then* $\alpha(G \cdot G) \ge n$.

Corollary 9.14 *For a self complementary graph* G, *we have* $\Theta(G) \ge \sqrt{n}$.

<u>PUZZLE</u>: Find a graph on 11 vertices which is not 3-colorable and has no triangles.
Definition: odd-girth(G) := the length of the shortest odd cycle in G.

Theorem 9.15 $(\forall k,\ell)(\exists G)(\chi(G) \ge k$ *and* $\text{odd-girth}(G) \ge \ell)$.

Proof Idea: Recall (Lecture 6, Theorem 6.5) Borsuk's Theorem: If $\mathbb{S}^n = A_0 \cup \cdots \cup A_n$ then $(\exists i)(\text{diam}(A_i) = 2)$, where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$, i.e, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$. For every n and $\epsilon > 0$, consider the graph $G_{n,\epsilon}$ defined as follows: $V = \mathbb{S}^n$. Adjacency is defined by $x \sim y$ iff $||x - y|| > 2 - \epsilon$.

Exercise 9.16 $(\forall n, \epsilon > 0)(\chi(G_{n,\epsilon}) \geq n + 2)$. *(Hint: Borsuk's Theorem.)*

Exercise 9.17 *Complete the proof of Theorem 9.16*

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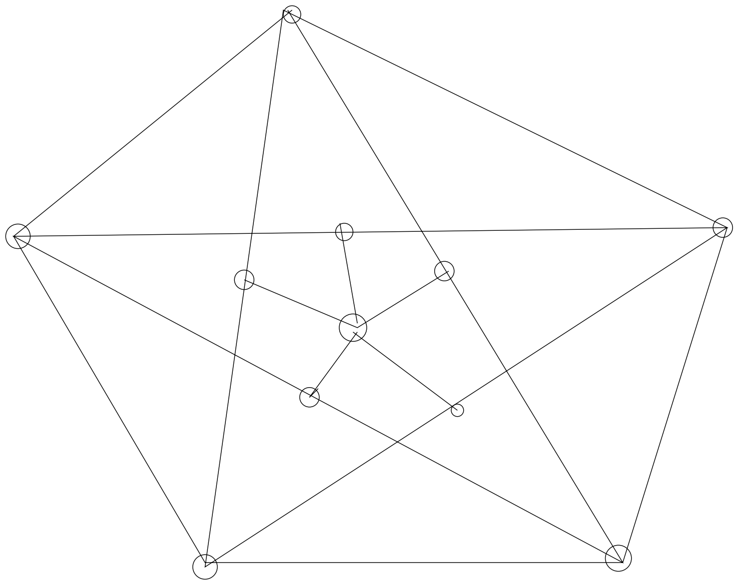

Figure 1: Grötzsch's graph: A graph on 11 vertices which is not 3-colorable and has no triangles.