Revised notes 4-18-2005

CMSC 27400-1/37200-1 Combinatorics and Probability	**Spring 2005**
Lecture 8: April 15, 2005	
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TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
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Chromatic number, degree, and independence number of graphs

Exercise 8.1 *If G is a graph with the maximum degree* $\deg_{\max}$ *then* $\chi(G) \leq 1 + \deg_{\max}$.

Exercise 8.2 *(a) If G is a directed graph such that the outdegree of every vertex in G is $\leq k$ and G has no* loops *then* $\chi(G) \leq 2k+1$. *(A* loop *is an edge (x, x).)*
(b) This bound is tight for all values of k.

Definition: A graph G is *planar* if there exists a plane drawing of G without intersection of edges. *Examples:* K_4, $K_{3,2}$ are planar. K_5, $K_{3,3}$ are not planar.
READING HOMEWORK: Study planar graphs from a Discrete Maths text (such as Rosen.)

Exercise 8.3 *If G is a planar graph then* $\chi(G) \leq 6$. *(Hint: Use the fact that for any planar graph, the minimum degree* $\deg_{\min} \leq 5$.*)*

Exercise 8.4 *If G is triangle-free then* $\chi(G) = O(\sqrt{n})$.

Exercise 8.5 (Erdős) *If $\mathcal{H}$ is a k-uniform hypergraph ($k \geq 2$) and the number of edges in $\mathcal{H}$ is $m \leq 2^{k-1}$ then* $\chi(\mathcal{H}) \leq 2$. *(Hint: Probabilistic Method.)*

Let $m(k)$ = minimum number of edges of a 3-chromatic k-uniform hypergraph. Note that the previous exercise implies that $m(k) > 2^{k-1}$.

Exercise 8.6 ***(Erdős)** *Show that* $m(k) < ck^2 2^k$. *(Hint: Probabilistic Method.)*

Definition: The independence number of a graph G, $\alpha(G)$:= the maximum size of an independent set in G.
Examples: $\alpha(K_n) = 1$; $\alpha(\overline{K}_n) = n$; $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ (C_n denotes the n-cycle).

Exercise 8.7 *Prove: for every graph G,* $\quad \alpha(G)\chi(G) \geq n$.

Let *Queen-graph* be the graph of queens on a chessboard defined as follows. For every cell A on the chessboard we have a vertex $v_A \in V(\textit{Queen-graph})$. So, there are 64 vertices. $v_A \sim v_B$ iff a queen on cell A threatens cell B.

Exercise 8.8 *Show that* $\alpha(\textit{Queen-graph}) = 8$.

Exercise 8.9 *(a) Define the* Rook-graph.
(b) Show that $\alpha(\text{Rook-graph}) = 8$.
(c) Count the number of ways one can put 8 *rooks on the chessboard so that no two threaten each other (i.e., count the maximum independent sets in the* Rook-graph*).*

Consider a 5×5 *toroidal* chessboard. Here, a king can move from the rightmost square in a row to the leftmost square in the row or its two vertical neighbours and vice versa. Also, a king can move from the topmost square in a column to the bottom square in the column or its two horizontal neighbours and vice versa. Also, a king can move from a corner to the opposite corner.

Exercise 8.10 *What is the maximum number of kings we can accommodate on the* 5×5 *toroidal chessboard such that no two kings threaten each other (i.e., they are not adjacent horizontally, vertically or diagonally) ?*

Exercise 8.11 (Open) *For an n-dimensional* $7 \times 7 \times \cdots \times 7$ *toroidal chessboard, what is* $\alpha(\text{King-graph})$*?*

Let $\alpha_n = \alpha(\textit{King-graph})$ for the n-dimensional $7 \times 7 \times \cdots \times 7$ toroidal chessboard. Here, the number of vertices in the *King-graph* is 7^n. Each vertex can be represented as $(x_1, \ldots, x_n)$ where each $x_i \in F_7$ (integers modulo 7). The king's move is to change each coordinate by at most 1 and at least one coordinate should be changed. So, degree of each vertex becomes $3^n - 1$.

Exercise 8.12 *Prove that* $\lim_{n\to\infty} {\alpha_n}^{1/n}$ *exists. (Hint. Use Exercise 8.13 below.)*

This limit is called Shannon Capacity of C_7. OPEN: Its value is not known.

Exercise 8.13 *Suppose* $a_n > 0$ *for all* n *and* $a_{k+\ell} \geq a_k a_\ell$ *for all* k *and* ℓ. *Prove:*
(a) $L := \lim_{n\to\infty} {a_n}^{1/n}$ *exists. (The limit may be finite or infinite.)* *(b) For all* n, $L \geq a_n^{1/n}$.

Exercise 8.14 *Define the "strong product"* $G \cdot H$ *of graphs* G *and* H *in such a way that a product of cycles should be a toroidal King-graph. Prove:* $\alpha(G \cdot H) \geq \alpha(G)\alpha(H)$.

Define the Shannon capacity of the graph graph G as $\Theta(G) = \lim_{n\to\infty}(\alpha(G^n))^{1/n}$.

Exercise 8.15 *Prove that this limit always exists.*

Exercise 8.16 *Prove:* $\sqrt{5} \leq \Theta(C_5) \leq 5/2$.

Exercise 8.17 *Prove: if the graph* G *is self-complementary (isomorphic to its complement) then* $\Theta(G) \geq \sqrt{n}$ *where* n *is the number of vertices.*

Theorem 8.18 (Lovász) $\Theta(C_5) = \sqrt{5}$.