CMSC 27400-1/37200-1 Combinatorics and Probability	**Spring 2005**

Lecture 7: April 11, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
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Explicit Ramsey Graphs

Erdős proved that $n \not\longrightarrow (1 + 2\log_2 n, 1 + 2\log_2 n)$, i.e, $\exists$ a graph on n vertices without a *homogeneous subset* (clique or independent set) of size $1 + 2\log_2 n$. The proof using Probabilistic Method proves only the *existence* of such a graph. No explicit construction of graphs verifying $n \not\longrightarrow ((\log n)^c, (\log n)^c)$ is known for any $c > 0$. No polynomial time algorithm to construct such graphs is known.
We have seen an easy explicit construction that verifies $n \not\longrightarrow (1 + \sqrt{n}, 1 + \sqrt{n})$. (What is the construction?))
Zsigmond Nagy (1973) gave an explicit Ramsey graph showing that $\binom{t}{3} \not\longrightarrow (t+1, t+1)$. This implies that, $n \not\longrightarrow (cn^{1/3}, cn^{1/3})$ where $n = \binom{t}{3} \sim \frac{t^3}{6}$, so $t \sim (6n)^{1/3}$ and therefore $c \sim 6^{1/3}$. (Refer Exercise 5.5)
(*No clique of size* $t+1$ follows from Fisher's Inequality while *no independent set of size* $t+1$ follows from the Oddtown Theorem.)

Theorem 7.1 (Frankl-Wilson, 1980) *Explicit construction verifies the relation* $(\forall \epsilon > 0)(\exists n_0)(\forall n \geq n_0)(n \not\longrightarrow (n^\epsilon, n^\epsilon))$.
The proof is the extension of idea used by Zsigmond Nagy. The proof depends on an extremal hypergraph inequalty, also supplied by Frankl and Wilson in the same paper.

Question: Given a set $\{\ell_1, \dots, \ell_s\}$ of s integers where $s \leq n/2$, suppose $A_1, \dots, A_m \subseteq [n]$ are sets such that $(\forall i)(|A_i| = k)$ and $(\forall i \neq j)|A_i \cap A_j| \in \{\ell_1, \dots, \ell_s\}$. Then what is the maximum possible value of m in terms of n and s?
For $s = 2$, K_n will give us $m \geq \binom{n}{2}$.
For $s = 3$, $K_n^{(3)}$ will give us $m \geq \binom{n}{3}$.
For an arbitrary s, $K_n^{(s)}$ will give us $m \geq \binom{n}{s}$.

Theorem 7.2 (D. K. Ray-Chaudhury–Richard M. Wilson, 1964) *Suppose* $A_1, \ldots, A_m \subseteq [n]$ *such that* $(\forall i)(|A_i| = k)$ *and* $|A_i \cap A_j| \in \{\ell_1, \ldots, \ell_s\}$. *Then* $m \leq \binom{n}{s}$.

Theorem 7.3 (Frankl-Wilson, 1980) *Suppose* $A_1, \ldots, A_m \subseteq [n]$ *such that* $(\forall i)(|A_i| = k)$ *and* $L = \{\ell_1, \ldots, \ell_s\}$
(a) $k \notin L \pmod{p}$, *i.e.,* $(\forall j)(k \not\equiv \ell_j \pmod{p})$ *where* p *is a prime.*
(b) $(\forall i \neq j)(|A_i \cap A_j| \in L \pmod{p})$, *i.e.,* $((\forall i \neq j)(\exists r)(|A_i \cap A_j| \not\equiv \ell_r \pmod{p}))$.
Then $m \leq \binom{n}{s}$.

Theorem 7.3 has an application in constructing Explicit Ramsey Graphs. Let $V = \binom{[2p^2-1]}{p^2-1}$ where p is a prime. For $A, B \in V$ we define $A \sim B$ iff $|A \cap B| \not\equiv -1 \pmod{p}$.
Claim: The above defined graph has no homogeneous subset of size $> \binom{2p^2-1}{p-1}$.
This will show that, $\binom{2p^2-1}{p^2-1} \not\longrightarrow (1 + \binom{2p^2-1}{p-1}, 1 + \binom{2p^2-1}{p-1})$.

Exercise 7.4 *Prove Theorem 7.3.*

Exercise 7.5 *Prove the Claim. (Hint: For proving that there is no clique of desired size use Frankl-Wilson Theorem (Theorem 7.3). For proving that there is no independent set of desired size use Ray-Chaudhury - Wilson Theorem (Theorem 7.2).)*