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CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

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TA SCHEDULE: TA sessions are held in Ryerson-255, Monday, Tuesday and Thursday 5:30–6:30pm.
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Please check the course website for corrections/announcements

Chromatic number of a hypergraph

Notation: Let $\mathcal{H} = (V, E)$ be a hypergraph where $E \subseteq 2^V := \{\text{subsets of } V\}$.
A *legal coloring* of $\mathcal{H}$ is a function $f : V \longrightarrow C$ (C is a set of colors), such that no edge is monochromatic, i. e., $(\forall e \in E)(|f(e)| \geq 2)$. For a legal coloring to exist it is necessary and sufficient that $(\forall e \in E)(|e| \geq 2)$.
Definition: A hypergraph $\mathcal{H}$ is *k-colorable* if $\exists$ a legal coloring of $\mathcal{H}$ using $\leq k$ colors.
Definition: The *chromatic number* of $\mathcal{H}$ is $\chi(\mathcal{H}) := \min\{k|\mathcal{H} \text{ is } k\text{-colorable}\}$.
Examples: (a) $\chi(K_n) = n$. (b) For $r \geq 2$, $\chi(K_n^{(r)}) = \lceil \frac{n}{r-1} \rceil$. (Why?)
Definition: Kneser's Graph is a graph denoted by $K(t, r)$ where $V = \binom{[t]}{r}$ (the set of all r-subsets of $[t] := \{1, \dots, t\}$) and $A \sim B$ (meaning A is adjacent to B) if and only if $A \cap B = \emptyset$. Note that a Kneser's Graph is interesting only when $t \geq 2r + 1$. If $t < 2r$ then $K(t, r)$ is the empty graph, i. e., it has no edges. If $t = 2r$ then $K(t, r)$ is a *perfect matching*, i. e., a 1-regular graph, i. e., a graph in which the degree of every vertex is exactly 1.

Exercise 6.1 *$K(5, 2)$ is the Petersen Graph. (See two drawings in the "Graphs and Digraphs" handout.)*

Theorem 6.2 (Lovász, 1975) $\chi(K(t, r)) = t - 2r + 2$.
(This was open for 50 years under the name "Kneser's Conjecture".)

Proof of the upper bound $\chi(K(t,r)) = t-2r+2$. $[t] = \{1,\dots,t\}$. Color all the r-subsets of $[t]$ containing 1 using color 1. Color all r-subsets of $[t]$ containing 2 but not containing 1 by color 2. Continue the same way until you have exhausted $t-2r+1$ colors. At this point, all the remaining (uncolored) r-subsets are subsets of teh remaining $2r-1$ elements of $[t]$ and therefore they form an independent set in Kneser's graph. So we can use just one more color to color all remaining vertices. Therefore, Kneser's graph is colorable using $t-2r+2$ colors. □

Definition: The *n-sphere* $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is defined as $\mathbb{S}^n = \{\underline{x} \in \mathbb{R}^{n+1} : ||x|| = 1\}$.
Definition: For $\underline{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$, the *Euclidean norm* $||x||$ is defined as $||x|| := \sqrt{\underline{x}\cdot\underline{x}} = \sqrt{\sum_{i=1}^n {x_i}^2}$. The distance of $\underline{x}$ and $\underline{y} \in \mathbb{R}^n$ is $||x-y||$.

Exercise 6.3 *If* $\mathbb{S}^1 = R \cup B$ *then* $(\forall\epsilon)(\exists x,y \in R$ *or* $x,y \in B$ *such that* $||x-y|| > 2-\epsilon)$

Definition: Given $A \subseteq \mathbb{R}^{n+1}$, the *diameter* of A is defined as $\operatorname{diam}(A) := \sup_{x,y\in A}\{||x-y||\}$.
Question (Borsuk, 1934): What is the minimum number of parts into which $\mathbb{S}^n$ must be divided such that each part has diameter < 2 ?

Exercise 6.4 *For* $\mathbb{S}^n$, $n+2$ *parts suffice.*

Theorem 6.5 (Borsuk's Theorem) *The minimum number of parts is* $n+2$, *i. e., if* $\mathbb{S}^n = A_0 \cup \dots \cup A_n$ *($n+1$ parts) then* $(\exists i)(\operatorname{diam}(A_i) = 2)$.

Theorem 6.6 (Borsuk's Lemma) *Let* $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$ *be a continuous function, then* $(\exists x \in \mathbb{S}^n)(f(x) = f(-x))$.

Exercise 6.7 *Prove that Borsuk's Lemma implies Borsuk's Theorem.*

Exercise 6.8 (****) *Prove that Borsuk's Theorem implies Kneser's Conjecture (Theorem 6.1).*

Definition: The *girth* of a graph is the length of the shortest cycle in it.
(The girth of a forest is infinite.)
Definition: The *odd-girth* of a graph is the length of the shortest odd cycle in it.
(The odd-girth of a bipartite graph is infinite.)

Exercise 6.9 *Using Kneser's Conjecture (Theorem 6.1), prove that* $(\forall k,\ell)(\exists$ *a graph with odd-girth* $> k$ *and* $\chi > \ell)$.

Exercise 6.10 *Construct a* 4*-chromatic graph on* 11 *vertices with no triangle. (Hint:* 5*-fold symmetry)*