Revised 4-7-05

CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 5: April 06, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Tuesday and Thursday 5:30–6:30pm.
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Extremal Set systems: the Linear Algebra method

Question: What is the maximum value of m such that there exists sets $A_1, \ldots, A_m \subseteq V$, $|V| = n$ and $(\forall i \neq j)(|A_i \cap A_j| = 1)$?

Theorem 5.1 (Erdős-de Bruijn) *If* $A_1, \ldots, A_m \subseteq V$*,* $|V| = n$ *and* $(\forall i \neq j)(|A_i \cap A_j| = 1)$ *then* $m \leq n$*. The inequality is tight, i.e.,* $m = n$ *is possible.*

Theorem 5.2 (Generalized Fisher Inequality) *Let* $r \geq 1$*. If* $A_1, \ldots, A_m \subseteq V$*,* $|V| = n$ *and*
$(\forall i \neq j)(|A_i \cap A_j| = r)$ *then* $m \leq n$*.*

(R. A. Fisher proved this result in the special case of "block designs" (1940). The method of proof below is due to R. C. Bose (1950). He proved it for uniform set systems, i.e., the case when $|A_1| = \cdots = |A_m|$*. The full result was proved by Majumdar (1956) using Bose's method.)*

Notation: Let $V = \{1, 2, \ldots, n\}$ and $A \subseteq V$ then v_A is an n-dimensional $(0,1)$-vector defined as follows: $v_A(i) = 1$ if $i \in A$; $v_A(i) = 0$ otherwise. v_A is called the *incidence vector* of A.
If $v = (a_1, \ldots, a_n)$ and $w = (b_1, \ldots, b_n)$ are two vectors in $\mathbb{R}^n$, we define their *dot product* as, $v \cdot w := \sum_{i=1}^{n} a_i b_i$.

Claim Under the assumptions of the Generalized Fisher Inequality, $v_{A_1}, \ldots, v_{A_n} \in \mathbb{R}^n$ are linearly independent, i.e., $(\forall \lambda_1, \ldots, \lambda_m \in \mathbb{R})(\sum_{i=1}^{m} \lambda_i v_{A_i} = 0 \Rightarrow \lambda_1 = \cdots = \lambda_m = 0)$.

Since, $\dim \mathbb{R}^n = n$, every set of linearly independent vectors in $\mathbb{R}^n$ has at most n members. This show that the Generalized Fisher Inequality follows immedietly from the Claim. (Babai-Frankl, Chapter 2, highly recommended for the linear algebra background.)

Proof of Claim: *Observations:* If $A, B \subseteq V$ then $v_A \cdot v_B = |A \cap B|$.
Specifically, $v_A \cdot v_A = |A \cap A| = |A|$. So, under the assumptions of Generalized Fisher Ineqality, we have, (for $i \neq j$ $v_{A_i} \cdot v_{A_j} = r$ and $v_{A_i} \cdot v_{A_i} = |A_i| =: k_i$.)

Definition: A set system $A_1, \ldots, A_m$ is a *sunflower* if $(\exists K)(\forall i \neq j)(A_i \cap A_j = K)$. K is called the *kernel* of the sunflower and K may be empty.

Exercise 5.3 *(a) Prove the Theorem 5.2 if the set-system is a sunflower.*
(b) Prove the Claim for sunflower.

Now, we may assume $\{A_1, \ldots, A_m\}$ is *not* a sunflower. Therefore, $k_j > r$ for all j. (Why?)
$v_{A_j} \cdot (\sum_{i=1}^m \lambda_i v_{A_i}) = \sum_{i=1}^m \lambda_i (v_{A_j} \cdot v_{A_i}) = (\sum_{i=1, i \neq j}^m \lambda_i r) + \lambda_j k_j = (\sum_{i=1}^m \lambda_i r) - \lambda_j r + \lambda_j k_j$.
Now, let $L := \sum_{i=1}^m \lambda_i$ and suppose $\sum_{i=1}^m \lambda_i v_{A_i} = 0$, then for $j = 1, \ldots, m$

$$0 = rL + \lambda_j (k_j - r). \tag{1}$$

So, by equation (1),

$$\lambda_j = \frac{-rL}{k_j - r}. \tag{2}$$

$$L = \sum_{j=1}^m \lambda_j = \sum_{j=1}^m \frac{-rL}{k_j - r}. \tag{3}$$

Case 1: $L = 0$. In this case, by equation (2), $(\forall j)(\lambda_j = 0)$.
Case 2: $L \neq 0$. In this case, dividing equation (3) by L,
$1 = -(\sum_{j=1}^m \frac{r}{k_j - r}) < 0$, a contradiction, proving that *Case 2* can not occur. $\square$

This technique is "The Linear Algebra Method" (Bose 1950).

Exercise 5.4 ("Odd-town Theorem") *If $A_1, \ldots, A_m \subseteq V$, $|V| = n$ and $(\forall i)(|A_i| = odd)$ and $(\forall i \neq j)(|A_i \cap A_j| = even)$ then $m \leq n$.*

Exercise 5.5 (Zsigmond Nagy, 1972) *Consider the following graph on $n = \binom{t}{3}$ vertices: $V = \binom{[t]}{3}$; for $A, B \in V$, define A to be adjacent to B if $|A \cap B| = 1$. Prove that this graph verifies $\binom{t}{3} \not\longrightarrow (t+1, t+1)$.*
Note that this is an explicit Ramsey Graph demonstrating $n \not\longrightarrow (cn^{1/3}, cn^{1/3})$.