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CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 4: April 04, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Tuesday and Thursday 5:30–6:30pm.
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Extremal Graph Theory

Question: What is the maximum number of edges of a graph with n vertices without K_3?

Notation: Let $\text{ex}(n, K_3)$ denote the maximum number of edges of a graph with n vertices without K_3. In general, let $\text{ex}(n, U)$ denote the maximum number of edges of a graph with n vertices without having a subgraph isomorphic to U. An *extremal graph* is one which has the optimum number of edges under the given constraint.
A *complete biparite graph* with parts of size a and b has $n = a+b$ vertices and $m = ab$ edges. It is denoted by $K_{a,b}$. (Cf. Graphs and Digraphs handout.)

Observations: (a) The 5-cycle demonstrates that $\text{ex}(5, K_3) \geq 5$.
(b) $K_{2,3}$ demonstrates that $\text{ex}(5, K_3) \geq 6$.

Theorem 4.1 (Mandel-Turán Theorem) *(a)* $\text{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$ *and*
(b) the only extremal graph is $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Proof Idea: Induction on n in steps of 2.
Base cases: $n = 1$, $n = 2$.
Now suppose $n \geq 3$.
Inductive Hypothesis: Assume that the result is true with $n-2$ in place of n.
Inductive step: Let G be a graph on n vertices. If G doesn't have any edges then we are done. Otherwise, pick an edge (u, v). Consider $G' = G \backslash \{u, v\}$ (we delete the vertices u, v and all edges incident with them). G' has $n-2$ vertices. Therefore, by the Inductive Hypothesis, $m' = E(G') \leq \frac{(n-2)^2}{4}$.

Since G doesn't have a triangle, u and v don't have any common neighbors. So there are at most $n-2$ edges from $\{u, v\}$ to $V(G')$.

Therefore, $m = |E(G)| \leq 1 + (n-2) + m'$
$\leq 1 + (n-2) + \frac{(n-2)^2}{4} = \frac{n^2}{4}$.

Exercise 4.2 *Show that* $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ *is the only extremal graph.*

Exercise 4.3 *Disprove:* $\exists n_0$, $\epsilon > 0$ *such that* $\forall n \geq n_0$ *if* $G \not\supseteq K_3$, G *not bipartite, then* $m \leq \frac{n^2}{4}(1-\epsilon)$.

Exercise 4.4 (*) *Prove:* $\exists n_0$, $C > 0$ *such that* $\forall n \geq n_0$ *if* $G \not\supseteq K_3$, G *not bipartite, then* $m \leq \frac{n^2}{4} - Cn$.

Exercise 4.5 (Turán's Theorem) *(a)* $\mathrm{ex}(n, K_4) = |E(K_{a,b,c})| \sim \frac{n^2}{3}$ *where* $a+b+c = n$ *and* $|\max\{a,b,c\} - \min\{a,b,c\}| \leq 1$. *(Hint: Use induction in steps of* 3.*)*
(b) In fact, this is the only extremal graph.
(c) Generalize (a) and (b) to any number r *instead of* 4. *State and prove Turán's Theorem for* $\mathrm{ex}(n, K_r)$.

Exercise 4.6 $\dfrac{\mathrm{ex}(n, C_4)}{n} \longrightarrow \infty$

Theorem 4.7 (Kővári, Sós, Turán) $\mathrm{ex}(n, C_4) < \frac{1}{2}(n^{3/2} + n)$.

Exercise 4.8 (Inequality between the arithmetic and the quadratic mean) *For real numbers* $x_1, x_2, \ldots, x_n$ *we have*

$$\sqrt{\frac{{x_1}^2 + \cdots + {x_n}^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n}. \tag{1}$$

The left-hand side is the quadratic mean*, the right-side is the* arithmetic mean*.*

Proof Idea: Consider a graph G on n vertices. Let N = number of paths of lenght 2 in G. Counting this quantity in two different ways and comparing the result is the key to the solution. Since G doesn't have 4-cycle, no two vertices have more than one common neighbour, each path of lenght 2 is uniquely determined by its endpoints. Therefore, $N \leq \binom{n}{2}$. On the other hand, counting the paths of length 2 by their middle points, $N = \sum_{y \in V} \binom{\deg(y)}{2}$. Since $\sum_{y \in V} \deg(y) = 2m$ ("Handshake Theorem") and by the inequality between the quadratic and the arithmetic mean, $\sum_{y \in V} \frac{(\deg(y))^2}{n} \geq \left(\frac{\sum_{y \in V} deg(y)}{n}\right)^2 = \left(\frac{2m}{n}\right)^2$, we have

$$\binom{n}{2} \geq \frac{1}{2}\left(\frac{(2m)^2}{n} - 2m\right). \tag{2}$$

Refer to Matoušek-Nešetřil, Section 6.3, for the exact evaluation of inequality (2). Here is how we evaluate it asymptotically. We may assume $n = o(m)$; therefore $2m = o(\frac{(2m)^2}{n})$ and the right hand side is $\sim \frac{(2m)^2}{n}$. The left hand side is $\sim n^2$. So $\frac{n^2}{2} \gtrsim \frac{(2m)^2}{n}$; therefore $m^2 \lesssim \frac{n^3}{4}$ and $m \lesssim \frac{n^{3/2}}{2}$. $\square$