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CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 3: April 1, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Tuesday and Thursday 5:30–6:30pm.
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Estimating Ramsey numbers: the Probabilistic Method

Definition: $r^{(2)}(N) := \max\{t : N \longrightarrow (t,t)\}$.
That is, $N \longrightarrow ((r^{(2)}(N), r^{(2)}(N))$ and $N \not\longrightarrow (1 + r^{(2)}(N), 1 + r^{(2)}(N))$.

Theorem 3.1 $N \longrightarrow (\frac{1}{2}\log_2 N, \frac{1}{2}\log_2 N)$. *In other words,* $r^{(2)}(N) \geq \frac{1}{2}\log_2 N$.

Proof Idea Follow the lines of the proof of Ramsey's Theorem for graphs (infinite version). We start with a vertex v_1. At least half of the remaining vertices will be joined to v_1 by an edge of the same color. We pick v_2 from these. Having chosen $v_1, v_2, \ldots, v_k$ we are left with $N/2^k$ vertices such that $(\forall i)$ all edges from v_i to v_j $(j > i)$ have the same color. We stop when $k = \log_2 N$ For at least half of them, the "right-going color" is the same. This induces the required monochromatic clique. □

Theorem 3.2 $N \not\longrightarrow (1 + \sqrt{N}, 1 + \sqrt{N})$. *In other words,* $r^{(2)}(N) \leq \sqrt{N}$.

Proof Idea Consider the disjoint union of $\sqrt{N}$ cliques of size $\sqrt{N}$. This is a subgraph of K_N. Color all edges in this subgraph red and all the edges in the complement blue. This coloring will not have a clique of size $1 + \sqrt{N}$ of either color. □

Theorem 3.2 turns out to be a very weak result. Indeed, Paul Erdős proved the following, much stronger bound:

Theorem 3.3 (Erdős 1950) $N \not\longrightarrow (1 + 2\log_2 N, 1 + 2\log_2 N)$. *In other words,*

$$r^{(2)}(N) \leq \lceil 2\log_2 N \rceil. \quad (1)$$

Corollary 3.4 $r^{(2)}(N) = \Theta(\log N)$.

To prove Theorem 3.3, Erdős gave a *non-constructive* proof of existence of a 2-coloring of K_N without homogeneous subsets (subsets which induce a monochromatic clique) of size $1 + 2\log_2 N$. This paper inaugurated his celebrated "Probabilistic Method," one of the most powerful techniques in combinatorics. Consider a random 2-coloring of $E(K_N)$. We prove that for $k \geq 1 + 2\log_2 N$,
$P(\exists \text{homogeneous clique of size } k) \longrightarrow 0$ as $N \longrightarrow \infty$. Note that it would suffice to show that the probability is less than 1.
Idea of proof: We have $|V| = N$. Consider $A \subseteq V$ such that $|A| = k$.

$$P(A \text{ is homogeneous}) = 2^{1-\binom{k}{2}}. \tag{2}$$

So, by the union bound,

$$P((\exists A \subset V)(|A| = k \text{ and } A \text{ is homogeneous})) < \binom{N}{k} 2^{1-\binom{k}{2}}. \tag{3}$$

Hence we proved an arithmetic condition for the Ramsey numbers:

$$\binom{N}{k} 2^{1-\binom{k}{2}} \leq 1 \Rightarrow N \not\longrightarrow (k,k). \tag{4}$$

Since $\binom{N}{k} \leq N^k/k!$, it suffices that we have

$$N^k/k! 2^{1-\binom{k}{2}} \leq 1 \tag{5}$$

That is,

$$N^k 2^{1-\binom{k}{2}} \leq k!/2. \tag{6}$$

It suffices, then, to have

$$N^k 2^{1-\binom{k}{2}} \leq 1 \tag{7}$$

$$\left(N 2^{-\frac{k+1}{2}}\right)^k \leq 1 \tag{8}$$

$$N 2^{-\frac{k+1}{2}} \leq 1 \tag{9}$$

which is equivalent to $k \geq 1 + 2\log_2(N)$.
Note that, in fact what we proved is that for $k \geq 1 + 2\log_2(N)$, we have

$$P((\exists A \subset V)(|A| = k \text{ and } A \text{ is homogeneous})) < 2/k! \tag{10}$$

□

Big Open Problem: Observe the factor of 4 (asymptotic) gap between the lower and upper bounds on $r^{(2)}(N)$ ($\frac{1}{2}\log N$ versus $2\log N$). Narrow the gap (reduce the number 4 to, say, 3.9999.

Definition: $\log^*(N) := \min\{k : 2^{2^{\cdot^{\cdot^{2}}}}\ (k \text{ times}) \geq N\}$.
$\log^* 2 = 1$.
$\log^* 3 = \log^* 4 = 2$.
$\log^* 5 = \cdots = \log^* 16 = 3$.
$\log^* 17 = \cdots = \log^* 65{,}536 = 4$.
$\log^*(65{,}537) = \cdots = \log^*(2^{65{,}536}) = 5$.
So, for all "reasonable" values of n, $\log^* n \leq 5$. Yet $\lim_{n\to\infty} \log^* n = \infty$.

Exercise 3.5 *(a) Show that proof given in class for $r^{(3)}(N)$ yields $r^{(3)}(N) \geq C\log^*(N)$ where C is a constant. (b) Modify the proof to yield $r^{(3)}(N) \geq C\log\log(N)$.*

Exercise 3.6 (Probabilistic upper bound) *Show that $r^{(3)}(N) \leq C'\sqrt{\log_2 N}$ where C' is a constant.*

Big Open Problem: Observe exponential gap between lower and upper bounds on $r^{(3)}(N)$ ($\log\log N$ versus $\sqrt{\log N}$). Narrow the gap.