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CMSC 27400-1/37200-1 Combinatorics and Probability **Spring 2005**

Lecture 2: March 30, 2005

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TA SCHEDULE: TA sessions are held in Ryerson-255, Tuesday and Thursday 5:30–6:30pm.
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Ramsey Theory Continued

"$\nexists$ complete disorder"

Theorem 2.1 (Ramsey's theorem for graphs) $(\forall \ell_1, \dots, \ell_k)(\exists R$ *such that (if* $n \geq R$ *then* $\forall$ *partition of* $E(K_n) = E_1 \dot\cup \dots \dot\cup E_k)(\exists j)(\exists S \subseteq V(K_n))(|S| \geq \ell_j$ *and* $(\forall i_1 \neq i_2 \in S)(\{i_1, i_2\} \in E_j)$

Complete 3-uniform hypergraph

A *hypergraph* is a tuple (V, E) where V is set of vertices and E is set of edges. An *edge* is an arbitrary subset of V.

Notation: If S is a set and $k \geq 0$ is an integer then a k-subset of S is a subset of size k and $\binom{S}{k}$ is the set of all k-subsets of S.

In 3-*uniform* hypergraph, every edge is a 3-subset. i.e. $E \subseteq \binom{V}{3}$. Graphs are 2-uniform hypergraphs. The *complete* 3-*uniform hypergraph* is $K_n^{(3)} := (V, \binom{V}{3})$. It has $\binom{|V|}{3}$ edges.

Theorem 2.2 (Ramsey's Theorem for 3-uniform hypergraph) $(\forall \ell_1, \dots, \ell_k)(\exists R)$ *such that (if* $n \geq R$*) then* $(\forall$ *partition of* $E(K_n^{(3)}) = E_1 \dot\cup \dots \dot\cup E_k)(\exists j)(\exists S \subseteq V(K_n^{(3)}))(|S| \geq \ell_j$ *and* $(\forall$ *distinct* $i_1, i_2, i_3 \in S)(\{i_1, i_2, i_3\} \in E_j)$

Theorem 2.3 (Ramsey's Theorem) $(\forall r, \ell_1, \dots, \ell_k)(\exists R)$ *such that (if* $n \geq R$*) then* $(\forall$ *partition of* $E(K_n^{(r)}) = E_1 \dot\cup \dots \dot\cup E_k)(\exists j)(\exists S \subseteq V(K_n^{(r)}))(|S| \geq \ell_j$ *and* $\binom{S}{r} \subseteq E_j)$

Theorem 2.4 (Ramsey's Theorem: infinite version) $(\forall r)(\forall$ *partition of* $E(K_\infty^{(r)}) = E_1 \dot\cup \dots \dot\cup E_k)$ $(\exists j)(\exists$ *infinite subset* $S \subseteq V(K_\infty^{(r)}))(\binom{S}{r} \subseteq E_j)$

If we prove the infinite version (Theorem 2.4) then the finite case (Theorem 2.3) follows by König path lemma. Theorem 2.2 is a special case of Theorem 2.3. Theorem 2.1 was proved in the previous lecture. We will prove Theorem 2.4. Theorem 2.2 and Theorem 2.3 will then follow.

Proof of Theorem 2.4 By induction on r. $r = 2$ case is Theorem 2.1.

Illustration of induction step for $r = 3$*:* Consider ${K_\infty}^{(3)}$. Pick a vertex and call it v_1. Now all the pairs in $\binom{V({K_\infty}^{(3)})\backslash\{v_1\}}{2}$ will get an induced coloring. The color of a pair $\{u, v\}$ will be the color of the triplet $\{v_1, u, v\}$. By induction hypothesis, $(\exists j_1)(\exists$ infinite set $S_1 \subseteq V\backslash\{v_1\})$ such that $(\forall T \in \binom{S_1}{2})(\{v_1\} \cup T \in E_j)$

Similarly, we can get j_2 and $S_2 \subseteq S_1\backslash\{v_2\}$. Proceeding this way, we have j_{i+1} and $S_{i+1} \subseteq S_i\backslash\{v_i\}$ such that $(\forall T \in \binom{S_{i+1}}{2})(\{i+1\} \cup T \in E_{j_{i+1}})$

Therefore, $(\exists$ infinite sequence of vertices $W = \{v_1, v_2, \dots\} \subseteq V)$ and $(\exists$ infinite sequence of colors $j_1, j_2, \dots$ such that $(\forall i_1 < i_2 < i_3)(\{i_1, i_2, i_3\} \in E_{j_{i_1}})$ i.e. the color of a triple depends only on its smallest element. Now, choose the color which occurs infinitely often among $j_1, j_2, \dots$. This will give you the required infinite subset.

Comment: Ramsey's theorem is a generalization of Pigeon Hole Principle (PHP). $r = 1$ case is in fact the PHP ! We could have taken that as our base case.

Ramsey numbers

The smallest value of $R(r, \ell_1, \dots, \ell_k)$ in Ramsey's Theorem for graphs (Theorem 2.1) is called the Ramsey number for $(r, \ell_1, \dots, \ell_k)$ and is denoted by $R^{(r)}(r, \ell_1, \dots, \ell_k)$. We omit (r) if $r = 2$. For example: $R(3, 3) = 6$ means $6 \longrightarrow (3, 3)$ and $5 \not\longrightarrow (3, 3)$

Exercise 2.5 *(a)* $10 \longrightarrow (3, 4)$ *(b)** $9 \longrightarrow (3, 4)$ *(c)* $8 \not\longrightarrow (3, 4)$.

Exercise 2.6 *(a)* $17 \longrightarrow (3, 3, 3)$ *(b)* $16 \not\longrightarrow (3, 3, 3)$ *(Hint: Use finite field $GF(16)$)*

These exercises show that $R(3, 3) = 6$, $R(3, 4) = 9$, and $R(3, 3, 3) = 17$.

Note. The previously posted version erroneously stated that $R(3, 4) = 10$.

Exercise 2.7 (Esther Klein (1932)) 5 *points in the plane, no three on a line* $\Rightarrow$ 4 *form a convex quadrilateral.*

Klein-Erdős(Question) True / False ? $(\forall k)(\exists n)(\forall n$ points in the plane, no three on a line $)(\exists$ convex k-gon among them)

Exercise 2.8 (George Szekeres (1932)) *"True." (Hint: use Ramsey's Theorem.)*

Klein, Erdős, and Szekeres were undergraduates at the time. Szekeres, a chemistry major, unaware of Ramsey's Theorem, rediscovered it to prove the result. Erdős then improved the bound, Erdős and Szekeres wrote a joint paper about it (1934), and Szekeres married Klein, after which Erdős dubbed the result "The Happy Ending Theorem."