CMSC-37110 Discrete Mathematics: Solutions to First Quiz
October 2005

This quiz contributes 6% to your course grade.

1. (5 points) In a well-shuffled deck of n cards, numbered 1 through n, what is the probability that cards #1 and #2 come next to each other (in either order)? Your answer should be an extremely simple expression; make it as simple as possible. Prove your answer.

   Answer. The denominator is $n!$. The numerator is $2(n-1)!$: throw out card #1, shuffle the remaining $n-1$ cards; and then stick #1 back in either before or after #2. So the probability is

   $$\frac{2(n-1)!}{n!} = \frac{2}{n}.$$

2. (1+6 points) We pick a random word (string) of length n over the alphabet $\{A, B, C, D, E\}$. (a) How large is the sample space? (b) What is the probability that each letter actually occurs in the word? Give a closed-form expression (no dot-dot-dots or summation signs). (It will not be a very simple expression; do your best.)

   Answer. (a) 5^n. (b) Let $R(A)$ be the event that our random word does not include the letter A. Define $R(B), \ldots, R(E)$ analogously. We are interested in the probability of the event $T = \overline{R(A) \cup \cdots \cup R(E)}$. By inclusion-exclusion $P(T) = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$ where S_k is the sum of the probabilities of the k-wise intersections of the events. Now $P(R(A)) = (4/5)^n$ because for each of the n positions, the probability that the letter in that position is not A is $4/5$ and these n events are independent. Similarly $P(R(B)) = \cdots = P(R(E)) = (4/5)^n$. Now $P(R(A) \cap R(B)) = (3/5)^n$ because for each position, the probability that neither A, nor B is written in that position is $(3/5)$ and these n events are independent. The same is the result for any of the $\binom{5}{2} = 10$ pairwise intersections. Analogously, each of the $\binom{5}{3} = 10$ triplewise intersections has probability $(2/5)^n$; each of the $\binom{5}{4} = 5$ quadruple intersections has probability $(1/5)^n$, and the intersection of all the 5 events is empty (the word cannot miss all letters). In summary, $S_0 = 1$; $S_1 = 5 \cdot (4/5)^n$; $S_2 = 10 \cdot (3/5)^n$; $S_3 = 10 \cdot (2/5)^n$; $S_4 = 5(1/5)^n$; $S_5 = 0$. So

   $$P(B) = \frac{5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5}{5^n}.$$

3. (6 points) What is the expected number of Kings in a poker-hand? Prove your answer. Make sure you give a clear definition of each of the random variables you introduce. 4 out of the 6 points go for the definition.

Answer. Let N be the number of Kings in the hand. Then $N = X_1 + \cdots + X_5$, where X_i is the indicator variable of the event A_i that the i-th card is King (in the order dealt). Therefore $E(N) = E(X_1) + \cdots + E(X_5)$. Now $E(X_i) = P(A_i) = 1/13$ (the probability that the i-th card is King is $4/52 = 1/13$). So $E(N) = 5/13$.

4. (5 points) If two events, A and B, are positively correlated, what can we say about their complements, $\overline{A}$ and $\overline{B}$? Prove your answer.

   Answer. The condition says $P(A \cap B) > P(A)P(B)$. Claim: $\overline{A}$ and $\overline{B}$ are also positively correlated. So we need to prove $P(\overline{A} \cap \overline{B}) > P(\overline{A})P(\overline{B})$.

   Proof. $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = 1 - P(A) - P(B) + P(A \cap B) > 1 - P(A) - P(B) + P(A)P(B)$ $= (1 - P(A))(1 - P(B)) = P(\overline{A})P(\overline{B})$.

5. (6 points) As a reward for her good deeds, Lucrezia gets to play the following game. She flips n coins. If k come up Heads, she wins 2^k ducats. What is her expected win? Give a simple closed-form expression (no dot-dot-dots or summation signs).

   Answer. $(3/2)^n$. Proof: The probability that exactly k coins come up heads is $\binom{n}{k} 2^{-n}$. Therefore Lucrezia's expected win is

$$2^{-n} \sum_{k=0}^{n} \binom{n}{k} 2^k = 2^{-n}(1+2)^n = (3/2)^n.$$

   (We used the Binomial Theorem for $(1+2)^n$.)

6. (1+3 points) Alice flips n coins and obtains X heads and $n - X$ tails. Bob repeats the experiment and obtains Y heads and $n - Y$ tails. (a) What is the size of the sample space for the combined experiment? (b) (BONUS PROBLEM) What is the probability that $X = Y$? Find a very simple closed-form expression (no summation or dot-dot-dots). Prove your answer. If you use a result proved in class, clearly state but do not prove that result.

   Answer. (a) $2^{2n} = 4^n$. (b) The probability that $X = k$ is $\binom{n}{k} 2^{-n}$. The same is the probability that $Y = k$; and this event is independent of the outcome of Alice's experiment. So the probability that $X = Y = k$ is $\left(\binom{n}{k}\right)^2 2^{-2n}$. The event $X = Y$ is the union of the $n+1$ disjoint events $X = Y = k$ ($k = 0, \ldots, n$), therefore

$$P(X = Y) = 4^{-n} \sum_{k=0}^{n} \left(\binom{n}{k}\right)^2 = \binom{2n}{n} 4^{-n}.$$

7. (2 points) (BONUS PROBLEM) Prove: the product of k consecutive integers is always divisible by $k!$.

Answer. Let the largest of the k consecutive integers be n. Then the product of the k consecutive integers in question is

$N := n(n-1)\dots(n-k+1)$. The claim is that N is divisible by $k!$. But $N/k! = \binom{n}{k}$ is an integer, so indeed $k!$ divides N.

8. (3 points) (BONUS PROBLEM) Let $S_n = \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{n}{3j}$. Prove: $\left| S_n - \frac{2^n}{3} \right| \leq \frac{2}{3}$.

(This problem is left as a challenge problem.)