CMSC-37110 Discrete Mathematics
SOLUTIONS TO FIRST MIDTERM EXAM
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This exam contributes 20% to your course grade.

1. (4+6+4+B4+6 points) Your answers to the questions below should be very simple closed-form expressions (no summation symbols or dot-dot-dots). **Prove your answers.** (a) Count the words (strings) of length $k+\ell$ over the 2-letter alphabet $\{A, B\}$ with k occurrences of A and ℓ occurrences of B. Your answer should be a very simple formula. (b) Count the words (strings) of length $k+\ell+m$ over the 3-letter alphabet $\{A, B, C\}$ with k occurrences of A and ℓ occurrences of B and m occurrences of C. (c) Count those words (strings) of length n over the English alphabet of 26 letters, $\{A, B, \ldots, Z\}$ (Capital letters only) which do not have the same letter in two consecutive places (e.g., "WOOD" or "AAAG" are not permitted but "WORD" or "GHGH" are). (d) (BONUS PROBLEM) Count those words (strings) of length n over the over the 2-letter alphabet $\{A, B\}$ which have no consecutive As. (E.g., "BAAB" and "AAAB" are not permitted but "ABBA" and "ABAB" are.) (e) Among the strings of length n counted in part (d) (no consecutive As), how many have exactly k occurrences of A?

   Answer.

   (a) $\binom{k+\ell}{k}$.
   Reason: We need to choose k out of the $k+\ell$ positions for the As.

   (b) *First answer:* $\binom{k+\ell+m}{k}\binom{\ell+m}{\ell}$
   Reason: First choose the k places for the As, then the ℓ places for the Bs from the remaining $\ell+m$ places.

   Second answer: $\frac{(k+\ell+m)!}{k!\ell!m!}$. Reason ("King Matthias' shepherd's method"): Label each A as $A_1, \ldots, A_k$ and similarly with the Bs and Cs. Now we have $k+\ell+m$ distinct symbols which can be arranged in $(k+\ell+m)!$ permutations (total number of "legs"). This is an overcount by a factor of $k!\ell!m!$, the number of ways we can permute the As between themselves, the Bs between themselves, and the Cs between themselves ("number of legs per sheep"). The "number of sheep" is the quotient. – DO: verify that the two answers give the same number.

   (c) $26 \cdot 25^{n-1}$

(d) Let a_n denote the number of words in question. We claim that $a_n = F_{n+2}$ (the $(n+2)$-nd Fibonacci number).

Lemma. For $n \geq 2$ we have $a_n = a_{n-1} + a_{n-2}$.

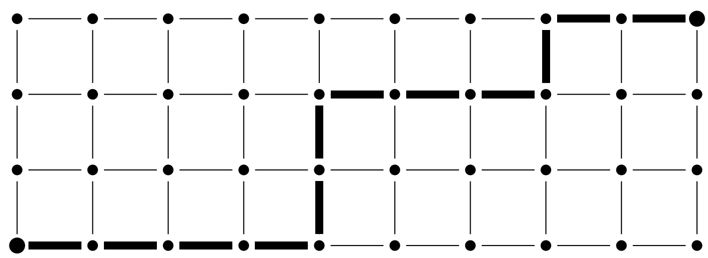
Proof. Let W_n denote the set of words of length n under consideration; so $a_n = |W_n|$. Now each word in W_n either begins with an a which is necessarily followed by a b; or it begins with a b:

$$W_n = abW_{n-2} \,\dot{\cup}\, bW_{n-1}.$$

Therefore $|W_n| = |W_{n-1}| + |W_{n-2}|$, proving the Lemma.
Now the statement follows by induction:
Bases cases: $a_0 = 1 = F_2$, $a_1 = 2 = F_3$.
Inductive Hypothesis: $a_k = F_{k+2}$ for all $k < n$.
Desired Conclusion: $a_n = F_{n+2}$.
Inductive step: We assume $n \geq 2$. By the Lemma we have $a_n = a_{n-1} + a_{n-2}$. By the Inductive Hypothesis, this is equal to $F_{n+1} + F_n$; by the Fibonacci recurrence, this is equal to F_{n+2}.

(e) Let us put down the $n-k$ copies of the letter B. This leaves $n-k+1$ "slots" for the k copies of A. We need to select which k of these to fill; the number of choices is $\binom{n-k+1}{k}$.

2. (6 points) Count the shortest paths from the bottom left corner to the top right corner of the $n \times k$ grid. (Note that the length of such a path is $n+k-2$). Your answer should be a very simple formula. Prove your answer. (The figure shows a 4×10 grid with a shortest path in question highlighted.)



Answer. Each path under consideration consists of $n-1$ steps UP and $k-1$ steps RIGHT, in any order. So we can encode the paths by strings of length $n+k-2$ over the 2-letter alphabet $\{UP, RIGHT\}$ with $k-1$ occurrences of RIGHT. According to Problem 1(a), the number of such strings is $\binom{n+k-2}{n-1}$.

3. (5 points) Let $a_n, b_n > 1$. Prove that the condition $a_n \sim b_n$ does NOT imply $a_n^n \sim b_n^n$. Before giving your counterexample, state clearly what properties your counterexample needs to have, and prove that it indeed has those properties.

Answer. We need two sequences $a_n, b_n > 1$ such that $a_n/b_n \to 1$ and $a_n^n/b_n^n \not\to 1$. Example: $a_n = \mathrm{e}^{2/n}$, $b_n = \mathrm{e}^{1/n}$. Now $a_n/b_n = \mathrm{e}^{1/n} \to \mathrm{e}^0 = 1$; but $a_n^n/b_n^n = \mathrm{e}^2/\mathrm{e} = \mathrm{e}$.

4. (2+4+4+B6 points) True or false? Prove your answers. (a) $\binom{n}{2} \sim n^2/2$. (b) $2^{\binom{n}{2}} \sim 2^{n^2/2}$. (c) $\ln \binom{n}{2} \sim 2\ln n$. (d) BONUS PROBLEM $(1+1/n)^{n^2} \sim \mathrm{e}^n$.

Answer.

(a) TRUE: $\dfrac{n(n-1)/2}{n^2/2} = \dfrac{n-1}{n} = 1 - \dfrac{1}{n} \to 1.$

(b) FALSE: $\dfrac{2^{n(n-1)/2}}{2^{n^2/2}} = 2^{-n/2} \to 0.$

(c) TRUE: we can use the result that if $a_n \sim b_n$ and $a_n \to \infty$ then $\ln a_n \sim \ln b_n$. By part (a) it follows that $\ln \binom{n}{2} \sim \ln(n^2/2) = 2\ln n - \ln 2 \sim 2\ln n$.

5. (2+3+B4 points) Evaluate each of the following sums as a closed-form expression (no summation symbols or dot-dot-dots). (a) $\sum_{k=0}^{n} 2^{-k}$.

(b) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^{-k}$. (c) (BONUS PROBLEM) Find the largest term in sum (b).

Answer.

(a) Sum of geometric series: $1 + q + \cdots + q^n = (1-q^{n+1})/(1-q)$. Substituting $q = 1/2$ we obtain $2 - 1/2^n$.

(b) This is the expansion of $(1+1/2)^n$ by the Binomial Theorem, so the answer is $(3/2)^n$.

(c) The quotient of two consecutive terms is

$$\frac{\binom{n}{k+1} 2^{-k-1}}{\binom{n}{k} 2^{-k}} = \frac{1}{2} \cdot \frac{n-k}{k+1}.$$

This is a monotone decreasing function of k; so the question is the smallest k such that this quotient is ≤ 1: $n - k \leq 2(k+1)$; in other words, $n - 2 \leq 3k$. So the largest term corresponds to $k = \lceil (n-2)/3 \rceil$.

6. (5 points) What is the probability that a poker hand is a "full house," i. e., 3 cards of the same kind and two other cards of the same kind (e. g., three 7s and two Kings). Give a simple closed-form expression; do not evaluate. (A poker hand consists of 5 cards from the standard deck of 52 cards.)

   Answer. $\dfrac{13 \cdot 12 \cdot 4 \cdot 6}{\binom{52}{5}}$. Explanation: The denominator is the number of poker hands. The numerator: $13 \cdot 12$ counts the ways to choose the two kinds involved (7, K in the above example); once these two kinds have been selected (in order), we have $\binom{4}{3} = 4$ choices for the three cards of the first kind (three 7s) and $\binom{4}{2} = 6$ choices for the pair of cards of the second kind (two Kings).

7. (3+2 points) (a) Define the statement that $\lim_{n\to\infty} a_n = -\infty$. Your answer should be a properly quantified formula ($\exists$, $\forall$) without English words. (b) Give an example of a sequence of numbers which has no limit (neither finite nor infinite).

   Answer. **(a)** $(\forall K)(\exists n_0)(\forall n > n_0)(a_n \le K)$. **(b)** $(-1)^n$.

8. (2+4+4+5+B5 points) Let $V = \{1, 2, \dots, n\}$, $n \ge 3$. Let us consider a random graph $\mathcal{G}$ on the vertex set V; adjacency is decided by coin flips. (a) What is the size of the sample space for this experiment? (b) Given $m \ge 0$, what is the probability that $\mathcal{G}$ has exactly m edges? (c) Let A_i denote the event that vertex i has even degree. What is the probability of A_i? (d) Decide whether A_1 and A_2 are positively or negatively correlated or independent. (e) BONUS PROBLEM. What is the probability that all vertices of $\mathcal{G}$ have even degree?

   Answer.

   (a) $2^{\binom{n}{2}}$.

   (b) $\dfrac{\binom{\binom{n}{2}}{m}}{2^{\binom{n}{2}}}$.

   (c) If $n = 1$ then $P(A_i) = 1$. If $n \ge 2$ then $P(A_i) = 1/2$. The reason is that if S is a nonempty set then half of its subsets are even and half are odd. (We had two proofs of this in class, one "combinatorial," the other by algebra (binomial theorem); REVIEW!) We apply this to the set S consisting of all vertices other than i; the neighbors of vertex i form a random subset of S.

   (d) The question implicitly assumes $n \ge 2$. If $n = 2$ then $A_1 = A_2$ (why?) so they are positively correlated. If $n \ge 3$ then we claim that A_1 and A_2 are independent. To prove this we need to show that $P(A_1 \cap A_2) = 1/4$. The trouble is that the degree of vertex

1 and the degree of vertex 2 both depend on a shared coin flip, namely the coin that decides whether or not 1 and 2 are adjacent. We use the Theorem of Complete Probabilities to get rid of this dependence. Let B be the event that 1 and 2 are adjacent. Then $P(B) = P(\overline{B}) = 1/2$ and $P(A_1 \cap A_2) = P(A_1 \cap A_2 \mid B)P(B) + P(A_1 \cap A_2 \mid \overline{B})P(\overline{B})$. Under the condition B, the events A_1 and A_2 are independent (having fixed the outcome of the coin flip for the edge $\{1, 2\}$ now makes these events depend only on disjoint sets of coin flips) so $P(A_1 \cap A_2 \mid B) = 1/4$ and similarly $P(A_1 \cap A_2 \mid \overline{B}) = 1/4$. Summarizing, $P(A_1 \cap A_2) = (1/4)(1/2) + (1/4)(1/2) = 1/4$.

(e) 2^{-n+1}. (Hint: The events $A_1, \ldots, A_{n-1}$ are independent, and A_n is determined by $A_1, \ldots, A_{n-1}$. Why?)

9. (1+8 points) (a) Define the diameter of a graph. (b) Prove: almost all graphs have diameter 2. First explain the exact meaning of this statement in terms of a limit relation.

Answer.

(a) Let $d(x, y)$ denote the length of the shortest path between vertices x, y. Then the diameter of the graph $G = (V, E)$ is $\min_{x,y \in V} d(x, y)$.

(b) Let p_n be the probability that a random graph on n vertices has diameter 2. The statement says that $\lim_{n\to\infty} p_n = 1$.

Proof. Fix two distinct vertices x and y. The probability that vertex z is a common neighbor of vertex x and vertex y is $1/4$; so the probability that z is not a common neighbor of x and y is $3/4$. For the $n - 2$ vertices z_i other than x, y, the events that z_i is not a common neighbor of x and y are indeoendent; therefore the probability that x and y have no common neighbor is $(3/4)^{n-2}$. Let $A(x, y)$ denote this event. The union of the $\binom{n}{2}$ events $A(x, y)$ is the event B that there exists a pair of vertices without a common neighbor. By the union bound, $P(B) < \binom{n}{2}(3/4)^{n-2}$. The right hand side goes to zero at an exponential rate ("exponential decay beats polynomial growth"). This means that in almost all graphs, every pair of vertices has a common neighbor. Therefore almost all graphs have diameter ≤ 2. To get the probability that the diameter is exactly 2, we need to subtract the probability that the diameter is 1, i. e., that the graph is complete. This probability is $2^{-\binom{n}{2}}$, which goes to zero very rapidly. Therefore almost all graphs have diameter exactly 2.

10. (4+4 points) Solve each of the following congruences or prove that no solution exists. (a) $5x \equiv 1 \pmod 7$ (b) $6x \equiv 1 \pmod{27}$.

Answer. Recall that $ax \equiv b \pmod{m}$ is solvable if and only if $\gcd(a, m) \mid b$. Therefore (a) is solvable and (b) is not. We can solve (a) by guessing; we quickly find that $x = 3$ is a solution. The complete solution therefore is: $x \equiv 3 \pmod{7}$ (the solution is unique modulo 7 because 5 and 7 are relatively prime). To review why (b) has no solution, assume $6x \equiv 1 \pmod{27}$. This implies that $6x \equiv 1 \pmod{3}$. But $6x \equiv 0 \pmod{3}$, so this would mean $0 \equiv 1 \pmod{3}$, a contradiction.

11. (4+4+5 points) (a) Count the 4-cycles in the complete graph K_n. A 4-cycle is a subgraph isomorphic to C_4. (Make sure your answer evaluates to 3 when $n = 4$.) (b) Count the 4-cycles in the complete bipartite graph $K_{r,s}$. (c) Let $\mathcal{G}$ be the random graph (as defined in a previous question). Determine the expected number of 4-cycles in $\mathcal{G}$. Make sure you give a clear definition of each random variable you introduce. 3 out of the 5 points go for this definition.

Answer.

(a) $3\binom{n}{4}$. Reason: there are $\binom{n}{4}$ ways to choose the set of 4 vertices of the cycle. The number of 4-cycles through a given set of 4 vertices is 3.

(b) $\binom{k}{2}\binom{\ell}{2}$. Reason: we need to pick two vertices from each part.

(c) Let $C_1, \dots, C_N$ be the 4-cycles of K_n where $N = 3\binom{n}{4}$ (by part (a)). Let X_i be the indicator of the event that C_i is part of $\mathcal{G}$, so $X_i = 1$ if $C_i \subseteq \mathcal{G}$ and $X_i = 0$ if $C_i \not\subseteq \mathcal{G}$. Let Y be the number of 4-cycles in $\mathcal{G}$; so $Y = X_1 + \cdots + X_N$. Therefore, by the linearity of expectation, $E(Y) = E(X_1) + \cdots + E(X_N)$. Now the expected value of an indicator variable is the probability of the event it indicates, so $E(X_i) = P(C_i \subseteq \mathcal{G}) = 2^{-4} = 1/16$ (because 4 coins must come up Heads to cause the event $C_i \subseteq \mathcal{G}$). To sum up, $E(Y) = N/16 = 3\binom{n}{4}/16$.

12. (3+B5 points) (a) State Fermat's little Theorem. (Do not prove.) (b) BONUS PROBLEM. Let $f(x) = 1 + x + x^2 + \cdots + x^{29}$. Prove: $(\forall x)(f(x) \equiv 0 \text{ or } \pm 1 \pmod{31})$.

Answer.

(a) If p is a prime and p does not divide the integer a then $a^{p-1} \equiv 1 \pmod{p}$.

(b) If $x \equiv 1 \pmod{31}$ then $f(x) \equiv 30 \equiv -1 \pmod{31}$. If $x \equiv 0 \pmod{31}$ then $f(x) \equiv 1 \pmod{31}$. For the remaining cases, observe that $(x-1)f(x) = x^{30} - 1$. Now $x \not\equiv 0 \pmod{31}$ and so by Fermat's little Theorem, $x^{30} - 1 \equiv 0 \pmod{31}$, so $(x-1)f(x) \equiv 0 \pmod{31}$. But $x - 1 \not\equiv 0 \pmod{31}$, so $f(x) \equiv 0 \pmod{31}$.

13. (3 points) A graph is *regular* of degree k if every vertex has degree k. Find a regular graph of degree 3 with 11 vertices or prove that no such a graph exists.

Answer. No such graph exists. Indeed, in such a graph, the sum of the degrees would be 33, an odd number. This would contradict the Handshake Theorem (the sum of the degrees is twice the number of edges).

14. (4 points) In a well-shuffled deck of n cards, numbered 1 through n, what is the probability that cards #1 and #2 come next to each other (in either order)? Your answer should be an extremely simple expression; make it as simple as possible. Prove your answer.

Answer. $2/n$. See the reason in the solution set to the First Quiz.

15. (1+5 points) We pick a random word (string) of length n over the alphabet $\{A, B, C, D, E\}$. (a) How large is the sample space? (b) What is the probability that all letters actually occur in the word? Give a closed-form expression (no dot-dot-dots or summation signs). (It will not be a very simple expression; do your best.)

Answer. See the solution set to the First Quiz.

16. (3 points) Prove: the product of k consecutive integers is always divisible by $k!$.

Answer. See the solution set to the First Quiz.

17. BONUS PROBLEM (B6 points) Let $S_n = \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{n}{3j}$. Prove: $\left| S_n - \frac{2^n}{3} \right| \leq \frac{2}{3}$.

(This problem continues to be left as a challenge problem.)