CMSC-37110 Discrete Mathematics: Second Quiz
11-01-2005

Typos corrected in problem 1 and in the bonus problems (8b,c)

Name (print): ____________________________________

Show all your work. Do not use book or notes. Do not use separate sheets, write your answers in the space provided after each question. You may use a pocket calculator for basic arithmetic only (no binomial coefficients, etc.). If you are not sure you understand a problem properly, **ask the instructor.** The BONUS PROBLEMS are undervalued, do not solve them until you solved the regular problems.

This quiz contributes 6% to your course grade.

1. (2+3 points) Write the following statements as properly quantified formulas with no English words except IF, THEN, AND, PRIME: **(a)** "For all sufficiently large x, the quantity $\pi(x)$ is greater than $x/(2\ln x)$." **(b)** "All primes of the form $4k+1$ can be written as the sum of two squares." (Example: $29 = 4\cdot 7 + 1 = 5^2 + 2^2$.)

2. (3 points) Determine m such that the following statement is true: $(\forall a, b)(a \equiv b \pmod{m}$ if and only if $(a \equiv b \pmod 4)$ and $a \equiv b \pmod 6)))$.

3. (3 points) Consider the following system of congruences: "$x \equiv 4 \pmod{15}$ and $x \equiv 10 \pmod 6$." Find a solution or prove that none exists.

4. (5 points) Prove: if $\gcd(b, 65) = 1$ then $b^{12} \equiv 1 \pmod{65}$.

5. (5 points) Construct a sequence a_n such that $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} a_n^n = \infty$.

6. (4+4 points) Compare the following two statements about a sequence $\{a_n\}$ of real numbers:

   (a) $\lim_{n\to\infty} a_n = \infty$;

   (b) $(\exists n_0)(\forall n \geq n_0)(a_{n+1} > a_n)$.

   Prove that (b) is neither necessary nor sufficient for (a). Clearly divide your answer into two parts and indicate which part disproves necessity and which disproves sufficiency.

7. (5 points) Find the remainder of 10^{902} when divided by 13. Your answer should be a number between 0 and 12. Do not use a calculator; show all your work.

8. (4+6+6 points) **(a)** State the Prime Number Theorem (PNT). Define your notation. **(b)** (Bonus problem) Let $S(x) = \sum_{p\leq x} p$. Prove: $S(x) \sim x^2/(2\ln x)$. Use the PNT. **(c)** (Bonus problem) Assuming $(\forall x \geq 2)(\prod_{p\leq x} p > 1.1^x)$, prove that $\pi(x) = \Omega(x/\ln x)$. (The product is over primes. Do not use the PNT.)

9. (4+4 points) **(a)** Define the big-oh relation: What does it mean that $f(x) = O(g(x))$? Use a properly quantified formula; no English words. **(b)** Disprove: if $a_n \sim b_n$ then $2^{a_n} = O(2^{b_n})$.

10. (4 points) Count the 4-cycles in the complete graph K_n.