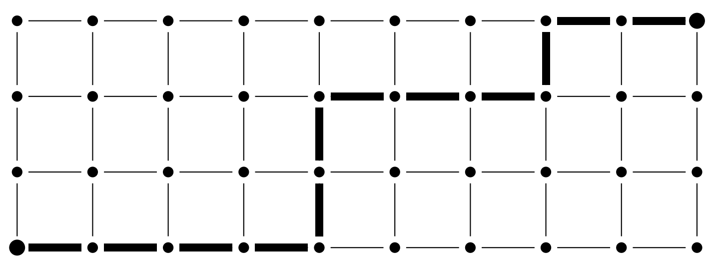
CMSC-37110 Discrete Mathematics
FIRST MIDTERM EXAM October 18, 2005

This exam contributes 20% to your course grade.

Do **not** *use book or notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as g.c.d. or binomial coefficients. **Show all your work.** If you are not sure of the meaning of a problem, **ask the proctors.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (4+6+4+B4+6 points) Your answers to the questions below should be very simple closed-form expressions (no summation symbols or dot-dot-dots). **Prove your answers.** (a) Count the words (strings) of length $k+\ell$ over the 2-letter alphabet $\{A, B\}$ with k occurrences of A and ℓ occurrences of B. Your answer should be a very simple formula. (b) Count the words (strings) of length $k+\ell+m$ over the 3-letter alphabet $\{A, B, C\}$ with k occurrences of A and ℓ occurrences of B and m occurrences of C. (c) Count those words (strings) of length n over the English alphabet of 26 letters, $\{A, B, \dots, Z\}$ (Capital letters only) which do not have the same letter in two consecutive places (e.g., "WOOD" or "AAAG" are not permitted but "WORD" or "GHGH" are). (d) (BONUS PROBLEM) Count those words (strings) of length n over the over the 2-letter alphabet $\{A, B\}$ which have no consecutive As. (E.g., "BAAB" and "AAAB" are not permitted but "ABBA" and "ABAB" are.) (e) Among the strings of length n counted in part (d) (no consecutive As), how many have exactly k occurrences of A?

2. (6 points) Count the shortest paths from the bottom left corner to the top right corner of the $n \times k$ grid. (Note that the length of such a path is $n+k-2$). Your answer should be a very simple formula. Prove your answer. (The figure shows a 4×10 grid with a shortest path in question highlighted.)



3. (5 points) Let $a_n, b_n > 1$. Prove that the condition $a_n \sim b_n$ does NOT imply $a_n^n \sim b_n^n$. Before giving your counterexample, state clearly what

properties your counterexample needs to have, and prove that it indeed has those properties.

4. (2+4+4+B6 points) True or false? Prove your answers. (a) $\binom{n}{2} \sim n^2/2$. (b) $2^{\binom{n}{2}} \sim 2^{n^2/2}$. (c) $\ln\binom{n}{2} \sim 2\ln n$. (d) BONUS PROBLEM $(1+1/n)^{n^2} \sim \mathrm{e}^n$.

5. (2+3+B4 points) Evaluate each of the following sums as a closed-form expression (no summation symbols or dot-dot-dots). (a) $\sum_{k=0}^{n} 2^{-k}$.
(b) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^{-k}$. (c) (BONUS PROBLEM) Find the largest term in sum (b).

6. (5 points) What is the probability that a poker hand is a "full house," i. e., 3 cards of the same kind and two other cards of the same kind (e. g., three 7s and two Kings). Give a simple closed-form expression; do not evaluate. (A poker hand consists of 5 cards from the standard deck of 52 cards.)

7. (3+2 points) (a) Define the statement that $\lim_{n\to\infty} a_n = -\infty$. Your answer should be a properly quantified formula ($\exists$, $\forall$) without English words. (b) Give an example of a sequence of numbers which has no limit (neither finite nor infinite).

8. (2+4+4+5+B5 points) Let $V = \{1, 2, \ldots, n\}$, $n \geq 3$. Let us consider a random graph $\mathcal{G}$ on the vertex set V; adjacency is decided by coin flips. (a) What is the size of the sample space for this experiment? (b) Given $m \geq 0$, what is the probability that $\mathcal{G}$ has exactly m edges? (c) Let A_i denote the event that vertex i has even degree. What is the probability of A_i? (d) Decide whether A_1 and A_2 are positively or negatively correlated or independent. (e) BONUS PROBLEM. What is the probability that all vertices of $\mathcal{G}$ have even degree?

9. (1+8 points) (a) Define the diameter of a graph. (b) Prove: almost all graphs have diameter 2. First explain the exact meaning of this statement in terms of a limit relation.

10. (4+4 points) Solve each of the following congruences or prove that no solution exists. (a) $5x \equiv 1 \pmod 7$ (b) $6x \equiv 1 \pmod{27}$.

11. (4+4+5 points) (a) Count the 4-cycles in the complete graph K_n. A 4-cycle is a subgraph isomorphic to C_4. (Make sure your answer evaluates to 3 when $n = 4$.) (b) Count the 4-cycles in the complete bipartite graph $K_{r,s}$. (c) Let $\mathcal{G}$ be the random graph (as defined in a previous question). Determine the expected number of 4-cycles in $\mathcal{G}$. Make sure you give a clear definition of each random variable you introduce. 3 out of the 5 points go for this definition.

12. (3+B5 points) (a) State Fermat's little Theorem. (Do not prove.) (b) BONUS PROBLEM. Let $f(x) = 1 + x + x^2 + \cdots + x^{29}$. Prove: $(\forall x)(f(x) \equiv 0 \text{ or } \pm 1 \pmod{31})$.

13. (3 points) A graph is *regular* of degree k if every vertex has degree k. Find a regular graph of degree 3 with 11 vertices or prove that no such a graph exists.

14. (4 points) In a well-shuffled deck of n cards, numbered 1 through n, what is the probability that cards #1 and #2 come next to each other (in either order)? Your answer should be an extremely simple expression; make it as simple as possible. Prove your answer.

15. (1+5 points) We pick a random word (string) of length n over the alphabet $\{A, B, C, D, E\}$. (a) How large is the sample space? (b) What is the probability that all letters actually occur in the word? Give a closed-form expression (no dot-dot-dots or summation signs). (It will not be a very simple expression; do your best.)

16. (3 points) Prove: the product of k consecutive integers is always divisible by $k!$.

17. BONUS PROBLEM (B6 points) Let $S_n = \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{n}{3j}$. Prove: $\left| S_n - \frac{2^n}{3} \right| \le \frac{2}{3}$.