Definition 1. The ordinary generating function of a sequence $a_n$ is:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Definition 2. The exponential generating function of a sequence $a_n$ is:

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

Generating functions are formal power series which form a ring under the natural addition and multiplication rules. For ordinary generating functions we get:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n)x^n$$

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} d_n x^n$$

where $d_n = \sum_{i=0}^{n} (n)_i a_i b_{n-i}$.

Similarly, we can add and multiply two exponential generating functions.

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (a_n + b_n)\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$$

where $d_n = \sum_{i=0}^{n} (\frac{n}{i}) a_i b_{n-i}$.

The last operation we will mention explicitly here is differentiation. For a generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, the derivative is $f'(x) = \sum_{n=0}^{\infty} na_n x^{n-1}$. For two generating functions $f$ and $g$, usual laws of derivation hold:

- $(f + g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $f(g(x))' = g'f'(g(x))$
0.1. **Determining a generating function from a recurrence.** Given a sequence $a_n$, we want to determine a closed form for a generating function of $a_n$.

**Example 1.** Let $a_n$ be the constant all ones sequence $(1, 1, 1, \ldots)$. Then the ordinary generating function for $a_n$ is $\sum_{n \geq 0} x^n$ which we know to be $\frac{1}{1-x}$.

Knowing certain basic relations such as in the last example will be very helpful in working with generating functions. In particular we note the following:

$$\frac{1}{(1-ax)^m} = 1 + \binom{m}{1} ax + \binom{m+1}{2} a^2 x^2 + \ldots + \binom{n+k-1}{k} a^k x^k + \ldots$$

**Example 2.** Let $a_n$ be defined by $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

In order to determine $f(x) = \sum_{n \geq 0} a_n x^n$, we start by expanding the first few terms of $f(x)$. Then we substitute in our recurrence on the coefficients. Finally we look for shifted versions of our original generating function.

$$f(x) = 1 + x + \sum_{n \geq 2} a_n x^n = 1 + x + \sum_{n \geq 2} (a_{n-1} + a_{n-2}) x^n = 1 + x + x \sum_{n \geq 2} a_{n-1} x^{n-1} + x^2 \sum_{n \geq 2} a_{n-2} x^{n-2} = 1 + x + x(f(x) - 1) + x^2 f(x) = \frac{1}{1-x-x^2}$$

0.2. **Recover a recurrence from a generating function.** Next we simply want to turn around the goal from the last section. Given the closed form of a generating function for a sequence, we would like to know the recurrence of the sequence.

If we are given the generating function as a rational function of polynomials, we use the technique of “equating coefficients on both sides”.

**Example 3.** Suppose we have a sequence $a_n$ whose ordinary generating function is:

$$f(x) = \frac{1 - x}{1 - 3x - x^2 + x^3}$$

Let $P(x) = 1 - x$ and $Q(x) = 1 - 3x - x^2 + x^3$. Then $Q(x)f(x) = P(x)$. For $n \geq 3$ we equate coefficients on both sides of this equation.

$$f(x)(1 - 3x - x^2 + x^3) = 1 - x$$

The coefficient of $x^n$ for $n \geq 3$ is 0 (as is easy to see from the right hand side). We must consider how from the left hand side, we contribute to the coefficient of $x^n$. We have the following terms: $(a_n x^n)1$, $(a_{n-1} x^{n-1})(-3x)$, $(a_{n-2} x^{n-2})(-x^2)$, and $(a_{n-3} x^{n-3})(x^3)$. These terms must add to zero, therefore $a_n - 3a_{n-1} - a_{n-2} + a_{n-3} = 0$, which gives a recurrence on our sequence. The only thing left to determine is $a_0$, $a_1$, and $a_2$. We can use the same method
as above for each of these terms. The constant term on the right hand side is 1, hence the constant term of \( f(x) \) must be 1. The coefficient of \( x \) is \(-1\) hence \((a_0)(-3) + (a_1)(1) = -1\). Since we already know \( a_0 = 1 \), we can solve to find \( a_1 = 2 \). Similarly we find \( a_2 = 7 \).

0.3. Solving a recursion. Here we show an example of how to use a generating function to solve a recursion. In particular, we are given a recursion but would like to have a non-recursive formula for the sequence.

**Example 4.** Suppose we are given the sequence defined by the recursion: \( a_n = 3a_{n-1} \). Let \( f(x) \) be its ordinary generating function.

\[
\begin{align*}
  f(x) &= a_0 + a_1 x + a_2 x^2 + \ldots \\
  3xf(x) &= a_0 3x + a_1 3x^2 + a_2 3x^3 + \ldots \\
  3x - 3xf(x) &= a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1)x^2 + \ldots \\
  f(x) - 3xf(x) &= a_0 \\
  \text{ (The relation } a_i - 3a_{i-1} = 0 \text{ is exactly our recurrence) } \\
  f(x) &= \frac{a_0}{1 - 3x} \\
  &= a_0(1 + 3x + 3^2x^2 + 3^3x^3 + \ldots) \\
  &= a_0 + 3a_0x + 3^2a_0x^2 + \ldots
\end{align*}
\]

Hence \( a_n = 3^n a_0 \).

0.4. Using generating functions in counting.

**Example 5.** Let \( t_n \) be the number of spanning trees of \( K_{2,n} \), the complete bipartite graph with \(|V_1| = 2\) and \(|V_2| = n\). We want to determine a formula for \( t_n \). First we set up a recursion. Clearly, \( t_1 = 1 \). For arbitrary \( n \), consider one vertex of \( V_2 \), say \( v \). If this vertex is connected to only one of the vertices in \( V_1 \), then the number of spanning trees is \( 2t_{n-1} \). If \( v \) is connected to both vertices of \( V_1 \), then the number of spanning trees is \( 2^{n-1} \) because no other vertex from \( V_2 \) can be connected to both vertices of \( V_1 \). Hence \( t_n \) satisfies \( t_n = 2t_{n-1} + 2^{n-1} \) for \( n \geq 2 \).

Now we solve our recursion.

Let \( t(x) \) be the ordinary generating function for \( t_n \).

\[
\begin{align*}
  t(x) &= t_0 + t_1 x + t_2 x^2 + \ldots \\
  2xt(x) &= 2t_0 x + 2t_1 x^2 + 2t_2 x^3 + \ldots \\
  t(x) - 2xt(x) &= t_0 + (t_1 - 2t_0)x + (t_2 - 2t_1)x^2 + \ldots \\
  t(x) - 2xt(x) &= x + 2x^2 + 2^2x^3 + \ldots \\
  t(x) - 2xt(x) &= \frac{x}{1 - 2x} \\
  t(x) &= \frac{x}{(1 - 2x)^2} \\
  t(x) &= x(1 + 2(2)x + 3(2^2)x^2 + \ldots) \\
  t(x) &= n2^{n-1}
\end{align*}
\]