CMSC-27100 Discrete Mathematics
FIRST MIDTERM EXAM October 27, 2004

This exam contributes 17% to your course grade.

Do **not** *use book or notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as g.c.d. **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.**

1. (4+4 points) Compare the following two statements about a sequence $\{a_n\}$ of real numbers:

   (a) $\lim_{n\to\infty} a_n = \infty$;

   (b) $(\exists n_0)(\forall n \geq n_0)(a_{n+1} > a_n)$.

   Prove that (b) is neither necessary nor sufficient for (a). Clearly divide your answer into two parts and indicate which part disproves necessity and which disproves sufficiency. In each part, clearly state the goal.

2. (3+3) True or false. Prove your answers.

   (a) $2^{\binom{n}{2}} \sim 2^{n^2/2}$

   (b) $\ln(n \ln n) \sim \ln n$

3. (3+3+5 points) Give closed-form expressions for the following sums:

   (a) $\sum_{k=0}^{n} 2^{k/2}$.

   (b) $\sum_{k=0}^{n} \binom{n}{k} 2^{k/2}$.

   (c) $\sum_{k=0}^{n} k\binom{n}{k}$. (Prove your answer.)

4. (3+4 points)

   (a) Prove: $\binom{2n}{n} > \dfrac{4^n}{2n+1}$. Do not use asymptotic formulas.

   (b) Prove: $\ln\binom{2n}{n} \sim cn^d$ for some constants c, d. State the values of c and d. Do not use the asymptotic formula for $\binom{2n}{n}$; instead, use the first part of this problem and the definition of asymptotic equality.

5. (6 points) Decide whether or not the following system of congruences is solvable. Prove your answer.

   $x \equiv 4 \pmod 8$

   $x \equiv 6 \pmod{10}$

   $x \equiv 1 \pmod 5$

6. (4 points) Find the remainder of $1! + 2! + \ldots + 99! + 100!$ when divided by 18.

7. (7 points) True or fale? Prove your answer. Do NOT use a calculator. Show all your work.

   "If g.c.d.$(x, 35) = 1$ then $x^{34} \equiv 1 \pmod{35}$."

8. (6 points) Let F_n denote the n-th Fibonacci number. (Recall that $F_0 = 0$ and $F_1 = 1$.) Prove: $(\forall n \geq 0)(F_n < 1.7^n)$. (Help: $17^2 = 289$.)

9. (2+4 points) Flip a coin n times. (a) What is the probability of getting exactly 2 heads? (b) What is the probability of getting at least 2 heads? Your answers should be simple closed-form expressions (no summation symbol, no "dot-dot-dots").

10. (5+2 points) Roll two 6-sided dice. (a) What is the probability that the first roll is 4 given that the sum of the dice is 8? (b) Let A be the event that the first roll is 4. Let B be the event that the sum of the dice is 8. Are events A and B independent, positively correlated, or negatively correlated?

11. (7 points) Let X_n be the number of of occurrences of the string "TEST" in a random string of length n over the English alphabet of 26 (upper case) letters. ("TEST" needs to appear as 4 consecutive letters, like in DCCTESTGHA.) For what value of n is $E(X_n) = 1$? Prove your answer.

12. (Challenge problem, no point value) Prove:
    $(\forall k \geq 1)(\exists x)(x^2 \equiv -1 \pmod{5^k}))$.

13. (Challenge problem, no point value) We have a deck of n cards, numbered $\{1, 2, \ldots, n\}$. A "hand" consists of k cards. What is the probability that a hand has no consecutive numbers? Your answer should be a very simple closed-form expression (a quotient of two binomial coefficients).