Discrete Mathematics – CMSC-37110-1 Homework 13 – November 22, 2005

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HOMEWORK. Please print your name on each sheet. Please try to make your solutions readable.
This homework is due next class, TUESDAY, NOVEMBER 29.
READ: the "Finite Markov Chains" handout (FMC); especially the Frobenius-Perron Theorem and its application to stationary distributions.
READ: the "Finite Probability Spaces" handout (FPS), all except Chapter 7.5 (Chernoff bound). Chap 7.5 is recommended but not required (may be used in a bonus problem)
RECOMMENDED READING: Linear Algebra: six lectures (also posted)
There are only DO problems at this time. Do not hand them in.

DO13.1 All problems from FMC, especially 8.1.24, 8.1.31, 8.1.32, 8.1.33, 8.1.35, 8.1.36, 8.1.37.

DO13.2 All problems from FPS, chapters 7.3 (Standard deviation) and 7.4 (Independent random variables), especially 7.3.4, 7.3.6, 7.4.4, 7.4.5, 7.4.6, 7.4.8, 7.4.14, 7.4.16.

DO13.3 We roll n dice. What is the expected value of the product of the numbers shown?

DO13.4 Let X_n be the number of triangles in a random graph. (a) Determine $E(X_n)$. (b) Determine $\mathrm{Var}(X_n)$. (c) Asymptotically evaluate $\mathrm{Var}(X_n)$. Show that $\mathrm{Var}(X_n) \sim an^b$ where a and b are constants. Determine a and b. (d) Compare the result with the sum Y_n of $\binom{n}{3}$ independent indicator variables with the same expectation (1/8). Is the variance less or more? (e) Use the result of (c) to prove the Weak Law of Large Numbers for X_n, i.e., prove that it is very likely that X_n stays close (within a factor of $(1 \pm \epsilon)$) to its expectation.

DO13.5 Prove that for a strongly connected digraph the following are equivalent:

(a) the number h divides the length of all cycles;
(b) the number h divides the length of all closed walks;
(c) it is possible to partition the vertex set into h classes $V_0, V_1, \ldots, V_{h-1}$ such that all edges go from V_i to V_{i+1} where the subscript is interpreted modulo h (i.e., $V_h = V_0$).

DO13.6 Prove: for an $n \times n$ matrix A, the following are equivalent:

(a) A has a left inverse ($A^{-1}A = I$)
(b) A has a right inverse ($AA^{-1} = I$)
(c) A is nonsingular ($\det(A) \neq 0$)
(d) zero is not an eigenvalue of A.

DO13.7 Prove: if A, B are $n \times n$ matrices and B is nonsingular then $B^{-1}A^tB = (B^{-1}AB)^t$.

DO13.8 Let A^T denote the *transpose* of A (replace $a_{i,j}$ by $a_{j,i}$). Prove: $(AB)^T = B^TA^T$.

DO13.9 Prove the triangle inequality, $||x + y|| \leq ||x|| + ||y||$, using the Cauchy-Schwarz inequality.

DO13.10 Recall that the operator norm of an $n \times n$ matrix A is defined as

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

where x ranges over all nonzero vectors in $\mathbb{R}^n$. Prove: (a) for all $x \in \mathbb{R}^n$, $||Ax|| \leq ||A|| \cdot ||x||$.

(b) For any two $n \times n$ matrices A, B we have $||AB|| \leq ||A|| \cdot ||B||$.

DO13.11 Recall the Spectral Theorem: If A is a symmetric real matrix then (a) all eigenvalues of A are real; (b) A has an orthonormal eigenbasis.

Use the Spectral Theorem to prove that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the symmertic real matrix A then $||A|| = \max_i |\lambda_i|$.

DO13.12 Recall that a real matrix B is *orthogonal* if $B^T = B^{-1}$. Prove: B is orthogonal if and only if $(\forall x \in \mathbb{R}^n)(||Bx|| = ||x||)$.

DO13.13 Let J be the $n \times n$ all-ones matrix (all entries are equal to 1). Let B be an orthogonal matrix of which the first column is $(1/\sqrt{n})\underline{1}$ where $\underline{1}$ denotes the all-ones column vector $(1, 1, \dots, 1)^T$. Prove: $B^{-1}JB$ is a diagonal matrix with $(1, 0, 0, \dots, 0)$ in the diagonal (in this order).

DO13.14 Consider the *rotation matrix*

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

(a) Prove: R_α is an orthogonal matrix.

(b) Compute the complex eigenvalues of R_α. (Note: the result is appealing.)

(c) Prove: $R_\alpha R_\beta = R_{\alpha+\beta}$.

(d) Prove: if λ is a (complex) eigenvalue of an $n \times n$ orthogonal matrix then $|\lambda| = 1$.