Discrete Mathematics – CMSC-37110-1 Homework 12 – November 17, 2005

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HOMEWORK. Please print your name on each sheet. Please try to make your solutions readable.

This homework is due next class, TUESDAY, NOVEMBER 22.

Please write solutions to challenge problems on a separate sheet.

READ: the "Finite Markov Chains" handout (FMC).

RECOMMENDED READING: Linear Algebra: six lectures (also posted)

The DO problems are for practice, do not hand them in.

DO12.1 Prove: (a) If a planar graph has no triangles then it has a vertex of degree ≤ 3. (b) Prove that if a planar graph has no triangles then it is 4-colorable.

DO12.2 (5+4 points) If a plane graph has $m = 3n - 6$ edges then every regions has 3 sides.

Homework (due at the beginning of class **Tuesday, Nov 22**).

HW12.1 (4+6+4 points) We define the Newtonian binomial coefficient $\binom{z}{k}$ as

$$\binom{z}{k} = \frac{z(z-1)\dots(z-k+1)}{k!}.$$

Here k is a nonnegative integer and z is an arbitrary real or complex number. If z is also a nonnegative integer, we shall use the term "ordinary binomial coefficient." Express the Newtonian binomial coefficients (a) $\binom{-n}{k}$ (b) $\binom{-1/2}{k}$ as simple expressions involving ordinary binomial coefficients only. Here n and k are nonnegative integers. (c) Asymptotically evaluate (b). Show that it is asymptotically equal to $b \cdot k^c \cdot d^k$ for constants b, c, d. Determine b, c, d.

Note. Newton's Binomial Theorem states that for any real or complex number z, for all x of absolute value $|x| < 1$,

$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k. \tag{1}$$

This theorem is useful in dealing with generator functions. Study and understand this theorem.

HW12.2 (10 points) Calculate the determinant of the $n \times n$ matrix $A_n = (a_{i,j})$ where $a_{i,i} = 1$ ($i = 1, \dots, n$), $a_{i,i+1} = -1$ ($i = 1, \dots, n-1$), and $a_{i,i-1} = 1$ ($i = 2, \dots, n$), all other entries are zero. The figure shows the matrix A_5. Hint: Let $d_n = \det(A_n)$. Experiment with small n; observe the pattern, make a conjecture. To prove the conjecture, expand by the last row to obtain a recurrence for the sequence d_n.

$$A_5 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

HW12.3 (6+1 points) (a) FMC 8.1.10 (find left and right eigenvectors) (b) FMC 8.1.14.

HW12.3 (8 points) FMC 8.1.11 (eigenvalues of stochastic matrix satisfy $|\lambda| \leq 1$)

HW12.3 (6 points) FMC 8.1.12 (left and right eigenvectors to distinct eigenvalues are orthogonal)