Discrete Mathematics – CMSC-37110-1 Homework 11 – November 15, 2005

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HOMEWORK. Please print your name on each sheet. Please try to make your solutions readable.

This homework is due next class, THURSDAY, NOVEMBER 17.

Please write solutions to challenge problems on a separate sheet.

READ: the ENTIRE "Graphs and Digraphs" handout (GD), especially the the section on Planarity (pp 55-59) and Digraph Terminology (pp60–65).

The DO problems are for practice, do not hand them in.

DO11.1 Prove: $F_n = \left\lceil \left(\frac{1+\sqrt{5}}{2} \right)^n \right\rfloor$. Here $\lceil x \rfloor$ denotes the integer nearest to x. (F_n is the n-th Fibonacci number.)

DO11.2 Prove: if G is a planar graph with no triangles and at least 3 vertices then $m \le 2n - 4$. (Here n is the number of vertices and m is the number of edges.) Hint: in class we proved $m \le 3n - 6$ for all planar graphs with at least 3 vertices. Modify the proof to take into account that there are no triangles.)

DO11.3 Prove: a plane tree has one region.

DO11.4 Study the Platonic solids as planar graphs (tetrahedron, cube, octahedron, dodecahedron, icosahedron). Verify Euler's formula for each.

DO11.5 What is the difference between a plane graph and a planar graph?

DO11.6 Prove: if a plane graph has n vertices, m edges, r regions, and c connected components then $n - m + r = c + 1$.

DO11.7 Draw a planar graph G such that every vertex has degree ≥ 2 but the graph is not Hamiltonian (has no Hamilton cycle).

DO11.8 Prove: every bipartite graph with ≤ 5 vertices is planar.

DO11.9 (a) Prove: if in a directed graph there is a directed walk from x to y then there is a directed path from x to y. (b) Prove that "accessibility" is a transitive relation on the vertices of a graph.

DO11.10 Prove: Mutual accessibility is an equivalence relation on the set of vertices of a digraph.

DO11.11 Prove: Accessibility defines a partial order on the set of strong components of a digraph.

DO11.12 Prove: a digraph can be topologically sorted if and only if it is a DAG (directed acyclic graph). (Hint: prove the following lemma: a DAG always has a vertex of out-degree zero.)

Homework (due at the beginning of class **Thursday, Nov 17**).

HW11.1 (5+8 points) Consider the sequence defined by the recurrence $b_0 = 7$, $b_1 = 16$, $b_n = 5b_{n-1} - 6b_{n-2}$. (a) Find a simple closed-form expression for the ordinary generating function $f(x) = \sum_{n=0}^{\infty} b_n x^n$.

(b) Use this expression to find a simple explicit formula for b_n (as a linear combination of two geometric progressions).

HW11.2 (6 points) Prove: every planar graph is 6-colorable. (Note: 6-colorability does not mean all 6 colors must actually be used. It only says 6 colors suffice.) (Hint. Use the result that every planar graph has a vertex of degree ≤ 5.)

HW11.3 (9 points) Prove: almost all graphs are not planar. (Hint: prove that it is exponentially unlikely that a given vertex has degree ≤ 5 in a random graph.)

HW11.4 (10 points) GD6.2.15 (if every vertex has the same in-degree as out-degree and the digraph is weakly connected then it is strongly connected)

HW11.5 (4 points) Prove: A DAG with n vertices has at most $\binom{n}{2}$ edges.

HW11.6 (10 points) Let G be a **strongly connected** digraph. Let a be the g.c.d. of the lengths of all cycles in G. Let x be a vertex and let $b(x)$ be the g.c.d. of the lengths of all closed walks in G, starting at x. Prove: $(\forall x \in V)(b(x) = a)$.