Discrete Mathematics – CMSC-37110-1 Homework 10 – November 10, 2005

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HOMEWORK. Please print your name on each sheet. Please try to make your solutions readable.

This homework is due next class, TUESDAY, NOVEMBER 15.

Please write solutions to challenge problems on a separate sheet.

READ: Review Chapter 2 of the Matoušek–Nešetřil text. Graphs and Digraphs handout, pp 45 – 59, except for the section on Planarity, from the bottom of page 55 to the top of page 59. Planarity will be the next topic, so you may want to read that section, too.

The DO problems are for practice, do not hand them in.

DO10.1 Prove: the graph G has a spanning tree if and only if G is connected.

DO10.2 Prove: (a) the sum of every row of the Laplacian of a graph is zero. (b) Therefore, the determinant of the Laplacian is zero.

DO10.3 Review Kirchhoff's Matrix Tree Theorem (Theorem 7.5.1 in the Matoušek – Nešetřil text). Cayley's formula from the Matrix Tree Theorem. Cayley's formula asserts that the number of spanning trees of the complete graph K_n is n^{n-2}.

DO10.4 Consider the degree constraint ("score") $1, 1, 2, 3, 2, 1$ on six vertices. (Vertices 1, 2, 6 must have degree 1, vertices 3, 5 must have degree 2, vertex 4 must have degree 3.) Calculate the number of trees satisfying this constraint using problem HW10.1. You will get 12 (verify!). Draw the 12 trees satifying the constraints.

DO10.5 Read the proof of Cayley's formula using the "Prüfer code" (Matoušek – Nešetřil, chapter 7.4).

DO10.6 Recall that the trinomial coefficient $\binom{n}{k,\ell,m}$ is defined as the coefficient of $x^k y^\ell z^m$ in the expansion of $(x+y+z)^n$. (Here $k, \ell, m \geq 0$ and $k+\ell+m = n$.) Prove:

$$\binom{n}{k,\ell,m} = \frac{n!}{k!\ell!m!}.$$

Hint: check the second solution to problem 1(b) in the First Midterm. (Count "number of sheep" by counting "legs," then dividing by number of "legs per sheep" – King Matthias' shepherd's principle.)

DO10.7 Generalize the preceding problem to multinomial coefficients (Matoušek–Nešetřil, Theorem 2.3.5).

DO10.8 (a) Prove: a connected graph has an Eulerian trail if and only if at most two of the vertices have odd degree. (b) Prove: a connected graph has a closed Eulerian trail if and only if all vertices have even degree.

DO10.9 Let $n, k \geq 2$. Prove: the $n \times k$ grid is Hamiltonian (has a Hamilton cycle) if and only if nk is even. (The proof of the "if" part should be a simple picture; the "only if" part has an AH-HA proof, one line.)

DO10.10 Prove: a graph G is bipartite if and only if G has no odd cycles. Use spanning trees for an elegant proof of the "if" part. The only if part is straightforward (why?).

DO10.11 (a) Prove: a positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. (b) Generalize to a rule of divisibility by $t-1$ in base-t number system. Prove. Make your proof simple by using congruences. (Hint: if the base-t digits of an integer N are $a_n, a_{n-1}, \ldots, a_0$ then the number is $N = \sum_{i=0}^{n} a_i t^i$. (Note that $0 \le a_i \le t-1$.) Determine $t^i \pmod{t}$.)

DO10.12 Given positive integers $d_, \ldots, d_n$ such that $\sum_{i=1}^{n} d_i = 2n-2$, prove that the number of trees on the vertex set $V = \{1, \ldots, n\}$ such that $\deg(i) = d_i$ is

$$\frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!}.$$

Hint: induction. NOTE: this problem was demoted from "BONUS HW" to "DO" problem because it is discussed in the text. (Check it out AFTER you gave it a decent try yourself.)

Homework (due at the beginning of class **Tuesday, Nov 15**).

HW10.1 (3+3 points) (a) Prove: a positive integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11. Example: $11 \mid 918,071$ because $9-1+8-0+7-1=22$ is divisible by 11. (b) Generalize to a rule of divisibility by $t+1$ in base-t number system. Prove. Make your proof simple by using congruences. (Hint: See DO10.11.)

HW10.2 (6 points) Count the terms in the Multinomial Theorem (expansion of $x_1+\cdots+x_m)^n$, see Matoušek–Nešetřil, Theorem 2.3.5. The number of terms is the number of m-tuples $(k_1, \ldots, k_m)$ of nonnegative integers such that $k_1+\cdots+k_m = n$. The answer will be a binomial coefficient of a very simple form; it will involve the parameters m and n. Prove your answer. Suggestion: solve it for $m=2$ and $m=3$ first to see a pattern. Solving it for $n=2$ (but variable m) may also be helpful.

HW10.3 (4 points) We distribute t chocolate bars among r children. Count the ways this can be done. Prove your answer. Example: if we have $r=9$ chocolate bars and $n=5$ children, we may write $9=3+0+2+1+3$, giving 3 chocolate bars to the first child, none to the second, 2 to the third, 1 to the fourth, and 3 to the fifth. (Hint: relate this question to the preceding problem.)

HW10.4 (4 points) Use the Multinomial Theorem and DO10.12 to prove Cayley's Formula.

HW10.5 (6 points) Suppose you recieve equal amounts of spam email and non-spam email. Further suppose the probability that a spam email contains the word "free" is $\frac{1}{3}$ and the probability that a non-spam e-mail contains the word "free" is $\frac{1}{30}$. Your software tells you that you recieved an e-mail that contains the word "free." What is the probability that the email is spam? Show all your work. – Instructions: let S denote the event that a random email item is spam, and F the event that a random email item contains the word "free." You are asked to calculate the conditional probability $P(S \mid F)$. You are given the probability $P(S)$ and the conditional probabilities $P(F \mid S)$ and $P(F \mid \overline{S})$. Identify the given values of each of these quantities . Give a general formula expressing $P(S \mid F)$ in terms of $P(S)$, $P(F \mid S)$ and $P(F \mid \overline{S})$ before plugging in the given values. Use the definition of conditional probability and the Theorem of Complete Probability.

In the subsequent problems, "GD" refers to the "Graphs and Digraphs" handout (expanded version, handed out on Nov 8). Changes made on Saturday, 11-12, 12:10am: Problem 10.10 was removed; each problem was given an identifying description to guard against misreading the problem number, typos, and possible confusion with the previous edition of the handout.

HW10.6 (2+2+4 points) GD 6.1.5 (self-complementary graphs)

HW10.7 (4+6 points) GD 6.1.6 (a) and (b) (number of non-isomorphic graphs)

HW10.8 (4 points) GD 6.1.10 (max number of edges of bipartite graph)

HW10.9 (4 points) GD 6.1.14 (count cycles of length k in K_n)

HW10.10 **ignore** (problem originally posted was identical with HW10.8)

HW10.11 (8 points) GD 6.1.25 (estimate number of non-isomorphic trees)

HW10.12 (2 points) GD 6.1.35 (bipartite graph, degrees, counting)

HW10.13 (8 points) GD 6.1.36 (3 random vertices in bipartite graph)

HW10.14 (4 points) GD 6.1.37 (find graph with many edges and no 5-cycle) (The answer should take less than one third of a line.)

HW10.15 (4 points) GD 6.1.38 (every graph is $1 + \deg_{\max}$-colorable)

HW10.16 (5 points) GD 6.1.40 (chromatic vs independence numbers)

HW10.17 (12 points) GD 6.1.42 (4-chromatic graph without triangles)

HW10.18 (4+4 points) GD 6.1.50 (Hamilton path, knights on 4×4 chessboard)

HW10.19 (5+0.5+0.5+3) GD 6.1.54 (regular graphs of girth 5)