Discrete Mathematics – CMSC-37110-1 Homework 8 – October 27, 2005
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HOMEWORK. Please print your name on each sheet. Please try to make your solutions readable. **This homework is due THURSDAY, NOVEMBER 3.**

READ: "Asymptotic notation" handout.

DO8.1 Suppose $f(x)$ and $g(x)$ are polynomials. Prove: (a) If $\deg(f) \le \deg(g)$ then $f(x) = O(g(x))$; (b) if $\deg(f) = \deg(g)$ then $f(x) = \Theta(g(x))$.

DO8.2 Prove: If $f(x) = o(g(x))$ then $f(x) = O(g(x))$.

DO8.3 Prove: If $f(x) \sim g(x)$ then $f(x) = \Theta(g(x))$.

DO8.4 (a) Prove: If $L =: \lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and $0 < |L| < \infty$ then $f = \Theta(g)$. (b) Exhibit two positive functions f and g such that $f = \Theta(g)$ but $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.

DO8.5 Prove: $4^n \ne O(3^n)$.

DO8.6 Prove: If $f(x) \sim g(x)$ and $f(x) = o(g(x))$ then $f(x) = g(x) = 0$ for all sufficiently large x (i. e., $(\exists x_0)(\forall x \ge x_0)(f(x) = g(x) = 0)$).

DO8.7 Prove: The Θ relation is an equivalence relation on the set of functions with same unbounded domain. This includes the case of sequences, i. e., functions whose domain is the set of nonnegative integers.

DO8.8 Prove: If $0 < f(x) \lesssim g(x)$ then $f = O(g)$.

Homework (due at the beginning of the class **next Thursday, Nov 3**):

HW8.1 (5+3 points) (a) Prove: If $f(x) = \Theta(g(x))$ and $\lim_{x\to\infty} f(x) = \infty$ then $\ln f(x) \sim \ln g(x)$. (b) Disprove: If $f(x) = \Theta(g(x))$ and $f(x) \ge 2$ and $g(x) \ge 2$ then $\ln f(x) \sim \ln g(x)$.

HW8.2 (3 points) Find two sequences, $a_n > 1$ and $b_n > 1$ such that neither $a_n = O(b_n)$ nor $a_n = \Omega(b_n)$ holds.

HW8.3 (8 points) Use the Prime Number Theorem to prove that $p_n \sim n \ln n$ where p_n denotes the n-th prime number.

HW8.4 (3 points) Prove: If $f(x)$ and $g(x)$ are polynomials of degree ≥ 1 with positive leading coefficients then $\ln f(x) = \Theta(\ln g(x))$. (A polynomial of degree n is an expression of the form $f(x) = \sum_{i=0}^{n} a_k x^k$ where the *leading coefficient* a_n is not zero.)

HW8.5 (2 points) Prove: $\binom{2n+1}{n} < 4^n$. (Reproduce the proof from class.)

HW8.6 (4+3+5 points) (a) Give a very simple proof of the following: $F_n = \Theta(F_{n+1})$ where F_n is the n-th Fibonacci number. The proof should be just a few lines, with no reference to irrational numbers such as the golden ratio. (b) Find an increasing sequence of positive numbers a_n such that $a_n \neq \Theta(a_{n+1})$. (c) Let b_n be a sequence of positive numbers. Prove: If $b_n = \Theta(b_{n+1})$ then $\ln b_n = O(n)$.

HW8.7 (Challenge problem) Prove that $n+1$ divides $\binom{2n}{n}$.

HW8.8 (Challenge problem) Recall that a *permutation* of a set A is a bijection $A \to A$. If f is a permutation then f^k is the composition of f with itself k times; so for instance $f^3(x) = f(f(f(x)))$. The *identity* is the permutation $id : A \to A$ defined by $id(x) = x$ for all $x \in A$. The *order* of f is the smallest positive k such that $f^k = id$. Let $r(n)$ be the largest order of permutations of a set of n elements. Prove: $\ln r(n) \gtrsim \sqrt{n \ln n}$. (Use the Prime Number Theorem.)