Discrete Mathematics – CMSC-37110-1 Homework 4 – October 6, 2005

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HOMEWORK. Please print your name on each sheet. Please try to make your solutions readable. Unless expressly stated otherwise, all solutions are due at the **beginning of the next class.**

Please write solutions to challenge problems on a separate sheet.

Handout: Chapers 1 and 2 from instructor's "Discrete Mathemtics" lecture notes (DM): "Logic" and "Asymptotic notation."

Reading: DM, pp. 1–10 (quantified formulas, asymptotic equality, inequality, little-oh notation). Review the concept of limits. Review precalculus, especially **logarithms**.

Do: (exercises you need to do but not hand in):

DO4.1 Let f, g be functions defined for all sufficiently large positive real numbers. Prove: $f \sim g$ if and only if $f - g = o(f)$.

(Recall: $f \sim g$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Comment on the function notation: f is a function and $f(x)$ is its value at x. Often, $f(x)$ is also used to denote the function itself, viewing x as an indeterminate rather than a number. So $f \sim g$ means the same thing as $f(x) \sim g(x)$.

DO4.2 Prove: $n(\ln n - 1) + (1/2)(\ln(2\pi) + \ln n) \sim n \ln n$.

DO4.3 Solve the problems of Quiz1, including the Bonus problems.

Homework (due at the beginning of the next class):

HW4.1 (3 points) Prove: $(\forall \epsilon > 0)\left(\lim_{x\to\infty} \frac{\ln x}{x^\epsilon} = 0.\right)$ Use algebra only; do not use calculus beyond the fact, proved in class, that $(\ln x)/x \to 0$ (and basic properties of limits).

HW4.2 (4 points) **(Exponential beats polynomial.)** Prove: for every polynomial $p(x)$ and every $C > 1$, we have $p(x) = o(C^x)$.

HW4.3 (4+3 points) (a) Prove: If $f \to \infty$ and $f \sim g$ then $\ln f \sim \ln g$. (b) Find functions f, g, defined for all positive real values of x, such that $(\forall x)(f(x) > 1 \text{ and } g(x) > 1)$ and $f \sim g$ but $\ln f \not\sim \ln g$.

HW4.4 (4 points) Prove the asymptotic relation $\ln(n!) \sim n \ln n$ from first prinicples, without using Stirling's formula. Use directly the definition of limits. Find a hint in DM (Ex. 2.3.11).

HW4.5 (2+4+3 points) Prove the following three statements. To prove parts (a) and (b), do not use Stirling's formula; make your proofs VERY simple. Use Stirling's formula for part (c).

(a) $\binom{2n}{n} < 4^n$

(b) $\binom{2n}{n} > \frac{4^n}{2n+1}$

(c) $\binom{2n}{n} \sim \frac{c4^n}{\sqrt{n}}$ for some constant c. Determine c.