RECURRENCES, GENERATING FUNCTIONS

Definition 1. *The* ***ordinary generating function*** *of a sequence* a_n *is:*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Definition 2. *The* ***exponential generating function*** *of a sequence* a_n *is:*

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

Generating functions are formal power series which form a ring under the natural addition and multiplication rules. For ordinary generating functions we get:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} d_n x^n$$

where $d_n = \sum_{i=0}^{n} a_i b_{n-i}$.

Similarly, we can add and multiply two exponential generating functions.

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (a_n + b_n) \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$$

where $d_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$.

The last operation we will mention explicitly here is differentiation. For a generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, the derivative is $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$. For two generating functions f and g, usual laws of derivation hold:

- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $f(g(x))' = g' f'(g(x))$

0.1. **Determining a generating function from a recurrence.** Given a sequence a_n, we want to determine a closed form for a generating function of a_n.

Example 1. Let a_n be the constant all ones sequence $(1, 1, 1, \ldots)$. Then the ordinary generating function for a_n is $\sum_{n\geq 0} x^n$ which we know to be $\frac{1}{1-x}$.

Knowing certain basic relations such as in the last example will be very helpful in working with generating functions. In particular we note the following:

$$\frac{1}{1-ax^m} = 1 + \binom{m}{1}ax + \binom{m+1}{2}a^2x^2 + \ldots + \binom{n+k-1}{k}a^kx^k + \ldots$$

Example 2. Let a_n be defined by $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

In order to determine $f(x) = \sum_{n\geq 0} a_nx^n$, we start by expanding the first few terms of $f(x)$. Then we substitute in our recurrence on the coefficients. Finally we look for shifted versions of our original generating function.

$$\begin{aligned} f(x) &= 1 + x + \textstyle\sum_{n\geq 2} a_nx^n \\ &= 1 + x + \textstyle\sum_{n\geq 2}(a_{n-1} + a_{n-2})x^n \\ &= 1 + x + x\textstyle\sum_{n\geq 2} a_{n-1}x^{n-1} + x^2\textstyle\sum_{n\geq 2} a_{n-2}x^{n-2} \\ &= 1 + x + x(f(x) - 1) + x^2f(x) \\ &= \tfrac{1}{1-x-x^2} \end{aligned}$$

0.2. **Recover a recurrence from a generating function.** Next we simply want to turn around the goal from the last section. Given the closed form of a generating function for a sequence, we would like to know the recurrence of the sequence.

If we are given the generating function as a rational function of polynomials, we use the technique of "equating coefficients on both sides".

Example 3. Suppose we have a sequence a_n whose ordinary generating function is:

$$f(x) = \frac{1-x}{1-3x-x^2+x^3}$$

Let $P(x) = 1 - x$ and $Q(x) = 1 - 3x - x^2 + x^3$. Then $Q(x)f(x) = P(x)$. For $n \geq 3$ we equate coefficients on both sides of this equation.

$$f(x)(1 - 3x - x^2 + x^3) = 1 - x$$

The coefficient of x^n for $n \geq 3$ is 0 (as is easy to see from the right hand side). We must consider how from the left hand side, we contribute to the coefficient of x^n. We have the following terms: $(a_nx^n)1$, $(a_{n-1}x^{n-1})(-3x)$, $(a_{n-2}x^{n-2})(-x^2)$, and $(a_{n-3}x^{n-3})(x^3)$. These terms must add to zero, therefore $a_n - 3a_{n-1} - a_{n-2} + a_{n-3} = 0$, which gives a recurrence on our sequence. The only thing left to determine is a_0, a_1, and a_2. We can use the same method as above for each of these terms. The constant term on the right hand side is 1, hence the constant term of $f(x)$ must be 1. The coefficient of x is -1 hence $(a_0)(-3)+(a_1)(1) = -1$. Since we already know $a_0 = 1$, we can solve to find $a_1 = 2$. Similarly we find $a_2 = 7$.

0.3. **Solving a recursion.** Here we show an example of how to use a generating function to solve a recursion. In particular, we are given a recursion but would like to have a non-recursive formula for the sequence.

Example 4. Suppose we are given the sequence defined by the recursion: $a_n = 3a_{n-1}$. Let $f(x)$ be its ordinary generating function.

$$\begin{array}{ccc}
f(x) & = & a_0 + a_1x + a_2x^2 + \ldots \\
3xf(x) & = & a_0 3x + a_1 3x^2 + a_2 3x^3 + \ldots \\
3x - 3xf(x) & = & a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1)x^2 + \ldots \\
f(x) - 3xf(x) & = & a_0 \\
 & & \text{(The relation } a_i - 3a_{i-1} = 0 \text{ is exactly our recurrance)} \\
f(x) & = & \frac{a_0}{1-3x} \\
 & = & a_0(1 + 3x + 3^2x^2 + 3^3x^3 + \ldots) \\
 & = & a_0 + 3a_0x + 3^2a_0x^2 + \ldots
\end{array}$$

Hence $a_n = 3^n a_0$.

0.4. **Using generating functions in counting.**

Example 5. Let t_n be the number of spanning trees of $K_{2,n}$, the complete bipartite graph with $|V_1| = 2$ and $|V_2| = n$. We want to determine a formula for t_n. First we set up a recursion. Clearly, $t_1 = 1$. For arbitrary n, consider one vertex of V_2, say v. If this vertex is connected to only one of the vertices in V_1, then the number of spanning trees is $2t_{n-1}$. If v is connected to both vertices of V_1, then the number of spanning trees is 2^{n-1} because no other vertex from V_2 can be connected to both vertices of V_1. Hence t_n satisfies $t_n = 2t_{n-1} + 2^{n-1}$ for $n \geq 2$.

Now we solve our recursion.

Let $t(x)$ be the ordinary generating function for t_n.

$$\begin{array}{ccc}
t(x) & = & t_0 + t_1x + t_2x^2 + \ldots \\
2xt(x) & = & 2t_0x + 2t_1x^2 + 2t_2x^3 + \ldots \\
t(x) - 2xt(x) & = & t_0 + (t_1 - 2t_0)x + (t_2 - 2t_1)x^2 + \ldots \\
t(x) - 2xt(x) & = & x + 2x^2 + 2^2x^3 + \ldots \\
t(x) - 2xt(x) & = & \frac{x}{1-2x} \\
t(x) & = & \frac{x}{(1-2x)^2} \\
t(x) & = & x(1 + 2(2)x + 3(2^2)x^2 + \ldots) \\
t(x) & = & n2^{n-1}
\end{array}$$