

## Discrete Math - Practice Problems - 2005

1. Let  $a = bq + r$  ( $a, b, q, r$  all positive integers).  
Prove that  $\gcd(a, b) = \gcd(b, r)$ .
2. Find the remainder of  $1! + 2! + \dots + 99! + 100!$  when divided by 18.
3. Give a combinatorial proof of the following identity:

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

4. Give a closed form expression for

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \dots$$

5. Count the number of 4 cycles in the complete bipartite graph  $K_{n,n}$ .
6. Let  $G$  be a graph on  $n$  vertices that is not connected. What is the maximum number of edges that  $G$  can have?
7. Suppose we color the *edges* of  $K_6$  with two colors. Prove that no matter how we color the edges there will exist a monochromatic triangle (A triangle of only one color). *Hint* Consider the degree of any vertex.
8. Let  $G$  be an arbitrary plane graph with  $v$  vertices,  $e$  edges,  $f$  regions, and  $m$  connected components. Prove that  $v - e + f = m + 1$ .
9. Does there exist a graph on 6 vertices which is (a) Hamiltonian but not Eulerian (b) Eulerian but not Hamiltonian?
10. Suppose you roll a fair 6 sided die 100 times. Let  $X$  be the number of times two consecutive rolls result in the same number. (a) What is the  $E[X]$ ? (b) What is the  $V(X)$ ?
11. Let  $X$  and  $Y$  be independent random variables. Prove that:  $V(X + Y) = V(X) + V(Y)$ .

12. Suppose a multiple choice problem has  $n$  possible answers. You guess an answer equally likely at random. If you get it wrong, you pick an answer equally at random from the remaining choices. You continue until you have guessed the right answer, each time only guessing from answers you have not tried yet. What is the expected number of guesses you take until you get the right answer?
13. Let  $a_n$  be the sequence for the Fibonacci numbers:  $a_0 = 0$ ,  $a_1 = 1$ , and  $a_n = a_{n-1} + a_{n-2}$  for  $n \geq 2$ . Find a simple closed expression for the ordinary generating function  $F(x) = \sum_{n \geq 0} a_n x^n$ .
14. If  $f(x)$  is the ordinary generating function for the sequence  $a_n$ , and  $b_n = na_n$  what is the generating function for  $b_n$  in terms of  $f(x)$ ?
15. Let  $a_n, b_n \rightarrow \infty$ . Show if  $a_n = \Theta(b_n)$  then  $\ln a_n \sim \ln b_n$ .