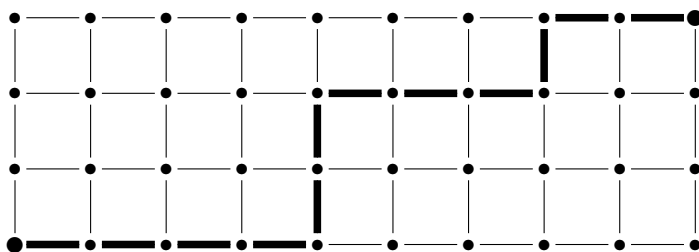


Homework 5 - Due Wednesday November 2nd

(1) Let $D(a)$ denote the set of divisors of the integer a . The congruence $ax \equiv b \pmod{m}$ is solvable if and only if $X \cap Y * W$ where X, Y, W are the sets $D(a), D(b), D(m)$ in some order and $*$ stands either for \subseteq or for \supseteq . Find the right relation between $D(a), D(b)$, and $D(m)$ (match up X, Y, W with $D(a), D(b), D(m)$, and state which inclusion is represented by $*$).

(2) Let $f(x) = 1 + x + x^2 + \dots + x^{29}$. Prove: $(\forall x)(f(x) \equiv 0 \text{ or } \pm 1 \pmod{31})$.

(3) Count the shortest paths from the bottom left corner to the top right corner of the $n \times k$ grid. A path consists of horizontal and vertical steps from one grid point to an adjacent grid point only. (Note that the length of such a path is $n + k - 2$). The picture shows a 4 by 10 grid and one shortest path.



(4) Out of n candidates, an association elects a president, two vice presidents, and a treasurer. Count the number of possible outcomes of the election.

(5) If $f(x)$ is the ordinary generating function for the sequence a_n , and $b_n = na_n$ what is the generating function for b_n in terms of $f(x)$?

(6) We roll a die n times. What is the probability that k out of the n times prime numbers comes up?

(7) Suppose you receive equal amounts of spam e-mail and non-spam e-mail. Further suppose the probability that a spam e-mail contains the word “free” is $\frac{1}{3}$ and the probability that a non-spam e-mail contains the word “free” is $\frac{1}{30}$. You receive an e-mail that contains the word “free” what is the probability that the e-mail is spam?

(8) Suppose you roll a fair 6 sided die 100 times. Let X be the number of times two consecutive rolls result in the same number. (a) What is the $E[X]$? (b) What is the $V(X)$?