# Algorithms CMSC-27000/37000 First Quiz. January 21, 2004

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**Name:** ________________________________ **U G** (circle one)

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. This quiz contributes 5% to your course grade.

1. (6 points) Write a pseudocode for binary search. Do NOT use recursive calls. You need to search for a real number $x$ in a sorted array $A$ of $n$ reals ($A[1] \leq A[2] \leq \ldots \leq A[n]$). Don't forget proper rounding. Write the pseudocode only; do not analyze.

2. (3 points) Give a formal definition of the Knapsack problem. Use mathematical expressions with very few English words; those words should be technical terms (e.g. input, real variable, constraint, etc.) (do NOT use words like weight, value, knapsack, etc.).

3. (3 points) For two sequences of real numbers, $\{a_n\}$ and $\{b_n\}$, define the relation $a_n \gtrsim b_n$ ("$a_n$ is greater than or asymptotically equal to $b_n$").

4. (3 points) Prove: the depth of a sorting network which sorts $n$ items must be $\gtrsim 2 \log n$. (log stands for $\log_2$.)

5. (G only, 2 points) Prove: $\log(n!) \sim n \log n$.

6. A divide-and-conquer algorithm reduces an instance (=input) of size $n$ to 3 instances of size $n/2$. (Ignore rounding.) The cost of the reduction is $O(n)$. Let $T(n)$ be the complexity of the algorithm on inputs of size $n$.

   (a) (1 point) Write a recurrenct inequality for $T(n)$.

   (b) (1 point) State the problem (input-output) solved in class by a an algorithm with the stated parameters. State the name of the discoverers of the algorithm.

   (c) (4 points) Ignore the cost of the reduction. Assuming $n = 2^k$, prove that $T(n) = O(n^\alpha)$ where $\alpha = \log 3$.

   (d) (G only, 4 points) Do not ignore the cost of the reduction. Prove that $T(n) = O(n^\alpha)$ where $\alpha = \log 3$. Use the method of reverse inequalities.

7. (a) (5 points) Let $X$ be an $n$-digit integer (decimal). We decide primality of $X$ (i. e., whether or not $X$ is a prime number) by trial division: we divide $X$ by each integer up to $\lfloor\sqrt{X}\rfloor$. Is this a polynomial-time algorithm? Prove your answer. (b) (G, only, 3 points) What if we only divide by the prime numbers (up to $\sqrt{X}$)?

8. (G only, 4 points) In the communication algorithm discussed in class, suppose $X = 213$ and $Y = 268$. Suppose Alice chooses her prime number $p$ at random from all primes with at most two *decimal* digits. What is the probability that Bob will erroneously report "$X = Y$"? (Use the fact that $\pi(100) = 25$ where $\pi(x)$ denotes the number of primes $\leq x$.) State what choices of $p$ will cause this error.