Homework 8

Due in class Monday October 25th. Ignore the point totals on the problems.

CMSC-27100 Discrete Mathematics
FIRST MIDTERM EXAM October 29, 2003
Show all your work. If you are not sure of the meaning of a problem, **ask the instructor.**

1. (6 points) Decide whether or not the following system of congruences is solvable. Prove your answer.

   $5x \equiv 2 \pmod 9$

   $x \equiv 7 \pmod{15}$

2. (4 points) Let $a_n, a_{n-1}, \ldots, a_1, a_0$ be the decimal digits of the positive integer N (a_0 is the lowest order digit). Prove: $N \equiv a_0 - a_1 + a_2 - + \ldots + (-1)^n a_n \pmod{11}$. *Hint.* Examine the sequence $10^k \pmod{11}$.

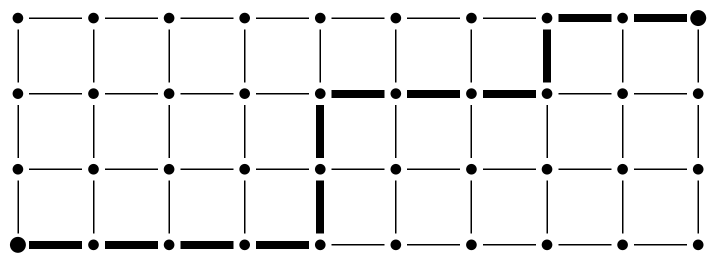
3. (6 points) Let $D(a)$ denote the set of divisors of the integer a. The congruence $ax \equiv b \pmod m$ is solvable if and only if $X \cap Y * W$ where X, Y, W are the sets $D(a), D(b), D(m)$ in some order and $*$ stands either for $\subseteq$ or for $\supseteq$. Find the right relation between $D(a), D(b)$, and $D(m)$ (match up X, Y, W with $D(a), D(b), D(m)$, and state which inclusion is represented by $*$). Prove your answer.

4. (6 points) Let $f(x) = 1 + x + x^2 + \ldots + x^{29}$. Prove: $(\forall x)(f(x) \equiv 0$ or $\pm 1 \pmod{31})$.

5. (4 points) Find a regular graph of degree 3 with 11 vertices or prove that such a graph does not exist.

6. (6 points) Suppose the graph G has n vertices, m edges, and no cycles. Prove that the number of connected components of G is $n - m$.

7. (6 points) Count the shortest paths from the bottom left corner to the top right corner of the $n \times k$ grid. (Note that the length of such a path is $n + k - 2$). Your answer should be a very simple formula. Prove your answer. (The figure shows a 4×10 grid with a shortest path in question highlighted.)



8. (4 points) We roll a die n times. What is the probability that k out of the n times prime numbers comes up? (A die has 6 sides, numbered 1 to 6.) Make your answer as simple as possible.

9. (2 points) Evaluate the sum $S_n = \sum_{i=0}^{\infty} \binom{n}{i} 2^{-3i}$. Your answer should be a very simple closed-form expression.

10. (4 points) We need to design a flag consisting of FOUR horizontal stripes. Each stripe has one color. Neighboring stripes have different colors. There are k colors available, including white. Only the two middle stripes can be white. Count the number of flags possible.

11. (a) (3 points) Let $a_n, b_n > 1$. Prove: $a_n \sim b_n$ does NOT imply $\ln a_n \sim \ln b_n$. (b) (4 points) Prove that under the additional condition that $a_n \geq 1.001$, the assumption $a_n \sim b_n$ DOES imply $\ln a_n \sim \ln b_n$.

12. (3 points) Consider the following "proof" of the obviously false statement

$$\lim_{n\to\infty} \frac{n}{n} = 0.$$

"Proof."

$$\frac{n}{n} = \frac{1}{n} + \ldots + \frac{1}{n}.$$

Every term on the right-hand side goes to zero, therefore their sum goes to $0 + \ldots + 0 = 0$.

What's wrong with this "proof"? Is it not true that if $a_n \to A$ and $b_n \to B$ then $a_n + b_n \to A + B$? Give a *brief* discussion.

13. (4 points) Count the cycles of length k in the complete graph K_n. (Note that for $k = 3$, your answer should specialize to $\binom{n}{3}$.)

14. (4 points) Let O_n denote the number of odd subsets of an n-set and E_n the number of even subsets of an n-set. For $n \geq 1$, prove that $O_n = E_n$. Give a bijective (combinatorial) proof. (Do NOT use binomial coefficients.)

15. (4 points) What is the probability that there is POKER in a poker hand? A POKER means 4 cards of the same kind (4 Kings, or 4 9s, etc.). Give a simple exact formula. (Recall that a “poker hand” is a set of 5 cards selected at random from the standard deck of 52 cards.)