4.1 Introduction

We recall Theorem 1.8:

**Theorem 4.1 (Complete Reducibility – Weyl)**

Every finite dimensional representation of $GL_n(\mathbb{C})$ admits a unique decomposition into irreducibles:

$$W = \bigoplus_i V_i^{m_i}, \quad (4.1)$$

Thus we wish to find the irreducible representations of $GL_n(\mathbb{C})$.

4.2 Irreducible Representations of $GL_n(\mathbb{C})$: First Construction (Deyruts)

Fix $\lambda$, a Young diagram of height $\leq n$. We associate with $\lambda$ an irreducible representation $V_\lambda$ (known as the Weyl module or Schur module). Let $x$ be a variable square matrix of size $n$, on which $GL_n(\mathbb{C})$ acts by multiplication on the left. This induces a $GL_n(\mathbb{C})$-action on $\mathbb{C}[x]$ by $(A \cdot f)(x) = f(A^t x)$. Hence $\mathbb{C}[x]$ is a representation of $GL_n(\mathbb{C})$ (infinite dimensional).

For any column $c = \begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix}$, where $c_i \in 1, ..., n$ are distinct, and $c_1 < c_2 < ... < c_l$, we define

$$e_c = \det \left( x^{(1...l)}_{c_1...c_l} \right),$$

i.e., the determinant of the minor of $x$ with columns $1$ to $l$ and rows $c_1$ to $c_l$.

For any semistandard tableau $T$ of shape $\lambda$, we define $e_T = \prod c_i e_c$, where $c$ ranges over the columns of $T$. Let $V_\lambda = \langle e_T \mid T \text{ semistandard of shape } \lambda \rangle$, i.e., the subspace of $\mathbb{C}[x]$ spanned by $e_T$’s. To show that $V_\lambda$ is invariant under $GL_n(\mathbb{C})$, it is enough to show that $A \cdot e_c = \sum' \alpha(c, c') \cdot e_c'$. To see that, we observe that $(A \cdot e_c)(x) = e_c(A^t x) = \det \left( (A^t x)^{(1...l)}_{c_1...c_l} \right) = \sum' \alpha(c, c') \cdot e_c'$, from the properties of the determinant. Note that the $c'$’s involved here are of the same shape as $c$ (i.e. same length). Since $A \cdot (fg) = (A \cdot f)(A \cdot g)$, it follows that $A \cdot e_T$ is also a linear combination of $e_T$, where the $T'$s have the same shape as $T$. Hence we have that $V_\lambda$ is also a representation of $GL_n(\mathbb{C})$.

**Theorem 4.2**

1. $V_\lambda$ is an irreducible representation of $GL_n(\mathbb{C})$ and also of $SL_n(\mathbb{C})$.

\footnote{We need $A'$ for it to be an action}
2. Every irreducible finite dimensional representation of $SL_n(\mathbb{C})$ is isomorphic to $V_\lambda$ for some $\lambda$ of height $< n$. Furthermore, $V_\lambda \ncong V_{\lambda'}$ if $\lambda \neq \lambda'$.

3. Every irreducible finite dimensional representation of $GL_n(\mathbb{C})$ is isomorphic to $V_\lambda \otimes \text{Det}^\alpha$, where $\text{Det}: A \mapsto \det(A)$ and $\alpha \in \mathbb{C}$. Similarly, $V_\lambda \otimes \text{Det}^\alpha \ncong V_{\lambda'} \otimes \text{Det}^{\alpha'}$ if $\lambda \neq \lambda'$ or $\alpha \neq \alpha'$.

4. The set $\{e_T | T \text{ semistandard of shape } \lambda\}$ is a basis of $V_\lambda$; i.e., the $e_T$’s are linearly independent.

**Proof:** (See Fulton & Harris, pp. 221-237.) Note that $e_T$ is defined for any $T$ of shape $\lambda$ (not necessarily semistandard). To prove 4. we write an arbitrary $R_T$ as a linear combination of semistandard $e_T$’s.

The above result shows that the Det representation is the only non-trivial representation of $GL_n(\mathbb{C})$ which when restricted to $SL_n(\mathbb{C})$ becomes trivial.

### 4.3 Characters

Let $W$ be a representation of $GL_n(\mathbb{C})$. Then define $\chi_W(g)$ to be the trace of $g$ on $W$, where $g \in GL_n(\mathbb{C})$ (This is a class function that is invariant on conjugacy classes). From elementary linear algebra, we know that every matrix in $GL_n(\mathbb{C})$ is conjugate to a matrix in Jordan canonical form. Hence $\chi_W$ is determined by its values on matrices in Jordan canonical form. It is easy to see (by looking at the Jordan canonical form or otherwise) that the diagonalizable elements form a dense subset of $GL_n(\mathbb{C})$, so for any rational representation (i.e., the entries of the representation are rational functions of the entries of the matrix), $\chi_W$ is determined by its values on diagonal matrices \[
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix},
\]
which we denote by $\chi_W(x_1,\ldots,x_n)$; this is called the character of $W$ (and is a rational polynomial function, with a suitable power of the determinant as its denominator in the case of $GL_n(\mathbb{C})$).

**Theorem 4.3** $\chi_{V_\lambda}(x_1,\ldots,x_n) = S_\lambda(x_1,\ldots,x_n)$, and if $\lambda \neq \lambda'$, then $S_\lambda \neq S_{\lambda'}$.

**Proof:**

Each $e_T$ is an eigenvector of \[
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix},
\]
where $\mu_i(T)$ is the number of $i$’s in $T$. We call $e_T$ a “weight vector” in the representation with weight $\mu = (\mu_1,\ldots,\mu_n)$. Since $\{e_T\}$ is a basis, we have

$$
\chi_W(x_1,\ldots,x_n) = \sum_T \prod_i x_i^{\mu_i(T)} = \sum_T \text{Content}(T),
$$

where we sum over semistandard $T$ of shape $\lambda$, and $\text{Content}(T) = S_\lambda(x_1,\ldots,x_n)$. ■
4.4 Irreducible Representations of $GL_n(\mathbb{C})$: Second Construction

Let $V = \mathbb{C}^n$, the standard representation of $GL_n(\mathbb{C})$. $GL_n(\mathbb{C})$ acts on $V^{\otimes d}$ from the left: $g(v_1 \otimes \cdots \otimes v_n) = g(v_1) \otimes \cdots \otimes g(v_n)$. $S_d$ acts on $V^{\otimes d}$ from the right: $(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$. The two actions commute, so $V^{\otimes d}$ is a representation of $GL_n(\mathbb{C}) \times S_d$.

Each irreducible representation of $GL_n(\mathbb{C}) \times S_d$ is isomorphic to one of the form $U \otimes U'$, where $U$ and $U'$ are irreducible representations of $GL_n(\mathbb{C})$ and $S_d$, respectively. This is in turn isomorphic to $V_\lambda \otimes S^\mu$, where $V_\lambda$ and $S^\mu$ are the corresponding Weyl and Specht modules, respectively. By Weyl’s Theorem (4.1), we have

$$V^{\otimes d} = \sum_{\lambda, \mu, \text{ht}(\lambda) \leq n, \text{size}(\mu) = d} m_{\lambda, \mu} V_\lambda \otimes S^\mu$$ (4.2)

Next we give an explicit decomposition of $V^{\otimes d}$. Fix $\lambda$, with size($\lambda$) = $d$. Fix $T$ to be any standard tableau of shape $\lambda$. Define the Young symmetrizer $c^T_\lambda$ to be $a_\lambda b_\lambda$, where $a_\lambda = \sum_{g \in \text{Row}(T)} e_g$ and $b_\lambda = \sum_{g \in \text{Col}(T)} \text{sgn}(g)e_g$. Let $S^T_\lambda(V) = V^{\otimes d} c^T_\lambda$, the image of $V^{\otimes d}$ under $c^T_\lambda$.

**Theorem 4.4** 1. $S^T_\lambda(V) \cong S^T_{\lambda'}(V)$ if $T$ and $T'$ are standard of the same shape $\lambda$. If $\text{ht}(\lambda) > n$, then $S_\lambda(V) = \{0\}$.

2. $S_\lambda(V)$ is isomorphic to $V_\lambda$ in Theorem 4.2; i.e., it is an irreducible representation of $GL_n(\mathbb{C})$ (if $\text{ht}(\lambda) \leq n$).

**Example 4.5** 1. $T = (1..d)$. We have $c_\lambda : v_1 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in S_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, so $S_\lambda = \text{Sym}^d(V)$.

2. $T = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. We have $c_\lambda : v_1 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in S_l} \text{sgn}(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, so $S_\lambda = \Lambda^l(V)$.

3. $\lambda = (2,1)$, $T = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$. $c_\lambda = c_{(2,1)} = (c_1 + c_{(1,2)})(c_1 - c_{(1,3)}) = 1 + c_{(1,2)} - c_{(1,3)} - c_{(1,2,3)}$.

Thus $S_\lambda(V) =$ the image of $c_\lambda$ on $V^{\otimes d} =$ the span of $v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2$.

**Theorem 4.6** 1. As a representation of $GL_n(\mathbb{C})$, $V^{\otimes d} = \sum_{\lambda, T} S^T_\lambda(V)$, where $\lambda$ ranges over Young diagrams of size $d$ and $T$ ranges over standard tableau of shape $\lambda$.

2. $V^{\otimes d} = \sum \lambda S_\lambda(V) \otimes S^\lambda(V)$, where again $\lambda$ ranges over Young diagrams of size $d$, and here the first $S$ term is the Weyl module and the second is the Specht module. In other words, in the formula (4.2), we have $m_{\lambda, \mu} = 1$ if $\mu = \lambda$, and $m_{\lambda, \mu} = 0$ if $\mu \neq \lambda$.

**Corollary 4.7** (to Theorem 4.3)
The finite dimensional representations of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are determined by their characters.
Proof: We have that \( S_\lambda \neq S_{\lambda'} \) if \( \lambda \neq \lambda' \) and that the \( S_\lambda \) are linearly independent. If \( V \) is any polynomial representation of \( GL_n(\mathbb{C}) \) (or \( SL_n(\mathbb{C}) \)) with character \( \chi_V(x_1, \ldots, x_n) \), then let (by Weyl’s theorem) \( v = \sum m_\lambda v_\lambda \). Then \( \chi_V = \sum m_\lambda S_\lambda \). Therefore to calculate \( m_\lambda \) we just express the character \( \chi_V \) in the Schur basis \( \{ S_\lambda \} \); the coefficients correspond to the multiplicities.

4.5 Tensor Products and a Decision Problem

Suppose we have \( W = S_\lambda(V) \otimes S_\mu(V) = V_\lambda \otimes V_\mu \). How does \( V_\lambda \otimes V_\mu \) decompose? Write \( V_\lambda \otimes V_\mu = \sum N_{\lambda \mu \nu} V_\nu \), so that now the problem is reduced to computing the terms \( N_{\lambda \mu \nu} \). We have \( \chi_W = S_\lambda S_\mu \), so \( S_\lambda S_\mu = \sum N_{\lambda \mu \nu} S_\nu \), which is a symmetric function, so we may express it in terms of the Schur basis. The Littlewood-Richardson rule gives a combinatorial property for computing the terms \( N_{\lambda \mu \nu} \). Using identities in symmetric function theory, they showed that \( N_{\lambda \mu \nu} \) is the number of ways in which the Young diagram \( \lambda \) can be expanded.

Problems

1. What is \( N_{\lambda \mu \nu} \) explicitly?
2. Given \( \beta \), is \( V_\beta \) a subset of \( V_\lambda \otimes V_\mu \), i.e., is \( N_{\lambda \mu \nu} \neq 0 \)?

Goal: A polynomial time algorithm for the decision problem “Does \( N_{\lambda \mu \nu} = 0 \)?".