A Brief Introduction to the Intuitionistic Propositional Calculus

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1 Introduction

For a classical mathematician, mathematics consists of the discovery of pre-existing mathematical truth. This understanding of mathematics is captured in Paul Erdös's notion of "God's Book of Mathematics," which contains the best mathematical definitions, theorems, and proofs, and from which fortunate mathematicians are occasionally permitted read a page.

Intuitionism takes the position that mathematical objects are mental constructions. Intuitionistic epistemology centers on *proof*, rather than *truth*. Thus, intuitionists analyze propositional combinations of mathematical statements in terms of what it takes to prove them, and a proof of $\varphi \wedge \psi$ consists of a proof of φ together with a proof of ψ, a proof of $\varphi \vee \psi$ consists of a proof of φ or a proof of ψ, while a proof of $\varphi \Rightarrow \psi$ consists of an algorithm that converts proofs of φ into proofs of ψ. For an intuitionist, a propositional formula is a tautology if it can be proven, e.g., $\alpha \Rightarrow \alpha$ is an intuitionistic tautology because to convert a proof p of α into a proof of α we simply return p.

The classical mathematician believes in the soundness of mathematical reasoning, i.e., that everything provable is true, and therefore all intuitionistic tautologies are also classical tautologies. The intuitionistic mathematician has no reason, however, to believe that everything that is true is provable (since he does not possess in his mind a construction that enables him to pass from an arbitrary but true mathematical statement ϕ to an intuitionistic proof of ϕ). Indeed, it is not difficult to give propositional formulae that are classical, but not intuitionistic, tautologies.

The standard example of a classical tautology that is not an intuitionistic tautology is the **law of the excluded middle**, $\alpha \vee \neg\alpha$. The law of the excluded middle is true classically because α must be either true or false, and $\alpha \vee \neg\alpha$ is true in either case. For the intuitionist, this is not satisfactory, as a proof of $\alpha \vee \neg\alpha$ must consist either of a proof of α or a proof of $\neg\alpha$, and argument given is neither.

One problem with this example, though, is that it depends on negation. Negation is a natural logical operator from a classical point of view, but it is not especially natural intuitionistically, as it is not immediately clear how we construct a negative, and as the treatment of the other logical connectives seems to require that somehow define a proof of $\neg\alpha$ in terms of a proof of α.

It is therefore interesting and relevant that there are negation-free propositional formulae that are classical, but not intuitionistic, tautologies. The simplest of these is **Peirce's Law**: $((\alpha \Rightarrow \beta) \Rightarrow \alpha) \Rightarrow \alpha$.

2 Intuitionistic Proofs

The intuitionistic propositional calculus is best organized as a natural deduction system.

2.1 The Positive Fragment

We'll begin by describing the natural, positive fragment of intuitionistic propositional logic.

Assumption

$$\Gamma, \alpha \vdash_I \alpha.$$

Conjunction Introduction

$$\frac{\Gamma \vdash_I \varphi \quad \Gamma \vdash_I \psi}{\Gamma \vdash_I \varphi \wedge \psi.}$$

Conjuction Elimination

$$\frac{\Gamma \vdash_I \varphi \wedge \psi}{\Gamma \vdash_I \varphi} \qquad \frac{\Gamma \vdash_I \varphi \wedge \psi}{\Gamma \vdash_I \psi}.$$

Disjunction Introduction

$$\frac{\Gamma \vdash_I \varphi}{\Gamma \vdash_I \varphi \vee \psi} \qquad \frac{\Gamma \vdash_I \psi}{\Gamma \vdash_I \varphi \vee \psi}.$$

Disjunction Elimination (Proof by Cases)

$$\frac{\Gamma \vdash_I \varphi \vee \psi \quad \Gamma \vdash_I \varphi \Rightarrow \xi \quad \Gamma \vdash_I \psi \Rightarrow \xi}{\Gamma \vdash_I \xi}$$

Implication Introduction (Deduction Theorem)

$$\frac{\Gamma, \varphi \vdash_I \psi}{\Gamma \vdash_I \varphi \Rightarrow \psi}$$

Implication Elimination (Modus Ponens)

$$\frac{\Gamma \vdash_I \varphi \Rightarrow \psi \quad \Gamma \vdash_I \varphi}{\Gamma \vdash_I \psi}$$

In such a presentation, proofs are naturally thought of as trees, where the leaves are instances of assumption, the internal nodes are instances of other rules of inference, and the root is the desired theorem. Thus:

$$\frac{\alpha \vdash_I \alpha}{\vdash_I \alpha \Rightarrow \alpha}$$

is a simple, but complete, proof. Typesetting proofs even slightly more complicated is a real trial, and therefore more complicated proofs are often presented in a linear style, like the following proof of **hypothetical syllogism**, $(\alpha \Rightarrow \beta) \Rightarrow [(\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)]$:

1. $\alpha, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash_I \alpha$,	assumption
2. $\alpha, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash_I \alpha \Rightarrow \beta$,	assumption
3. $\alpha, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash_I \beta$,	1, 2, implication elimination
4. $\alpha, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash_I \beta \Rightarrow \gamma$,	assumption
5. $\alpha, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash_I \gamma$,	3, 4, implication elimination
6. $\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash_I \alpha \Rightarrow \gamma$,	5, implication introduction
7. $\alpha \Rightarrow \beta, \vdash_I (\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)$,	6, deduction theorem
8. $\vdash_I (\alpha \Rightarrow \beta) \Rightarrow [(\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)]$,	7, deduction theorem

2.2 Negation

In order to understand the intuitionistic treatment of negation, it is useful to let $\perp$ denote a contradiction, i.e., a formula of the form $\alpha \wedge \neg\alpha$. We then introduce $\neg\varphi$ as an abbreviation for $\varphi \Rightarrow \perp$, which enables us to recast the implication introduction rule as a negation introduction rule:

Negation Introduction

$$\frac{\Gamma, \alpha \vdash_I \beta \quad \Gamma, \alpha \vdash_I \neg\beta}{\Gamma \vdash_I \neg\alpha}$$

Symmetry suggests that we need a negation elimination rule, too. The formula $(\neg\alpha \wedge \alpha) \Rightarrow \beta$ is a classical tautology that expresses the principle that everything follows from a contradiction[1]. We recast this in the equivalent form $\neg\alpha \Rightarrow (\alpha \Rightarrow \beta)$, and note that this is in fact intuitionistically valid! After all, if we have a proof of $\neg\alpha$, it follows that there cannot be a proof of α. Hence the empty function translates all (none) of the proofs of α into proofs of β. We then recast this formula as a rule of inference:

Negation Elimination

$$\frac{\Gamma \vdash_I \neg\alpha}{\Gamma \vdash_I \alpha \Rightarrow \beta}$$

We can give intuitionistic proofs of standard tautologies involving negation. First, **double negation introduction**, $\alpha \Rightarrow \neg\neg\alpha$.

1. $\alpha, \neg\alpha \vdash_I \neg\alpha$, assumption
2. $\alpha, \neg\alpha \vdash_I \alpha$, assumption
3. $\alpha \vdash_I \neg\neg\alpha$, 1, 2, negation introduction
4. $\vdash_I \alpha \Rightarrow \neg\neg\alpha$, 3, implication introduction

Double negation elimination, $\neg\neg\alpha \Rightarrow \alpha$, is a classic tautology, but it is *not* an intuitionistic tautology. Surprisingly, though, the special case of **triple negation reduction**, $\neg\neg\neg\alpha \Rightarrow \neg\alpha$, is an intuitionistic tautology:

1. $\neg\neg\neg\alpha, \alpha \vdash_I \neg\neg\alpha$, double negation introduction, 3
2. $\neg\neg\neg\alpha, \alpha \vdash_I \neg\neg\neg\alpha$, assumption
3. $\neg\neg\neg\alpha \vdash_I \neg\alpha$, 1, 2, negation introduction
4. $\vdash_I \neg\neg\neg\alpha \Rightarrow \neg\alpha$, 2, implication introduction

3 The Curry-Howard Isomorphism

There is a simple, and fruitful, analogy between the positive fragment of intuitionistic logic and the basic type constructors used in computer programming. The idea is that $\alpha \wedge \beta$ corresponds to a product (record) type,

[1] Bertrand Russell, a noted logician, philosopher, and atheist, once noted, "Grant me a contradiction, and I'll prove that I'm the pope!"

$\alpha \vee \beta$ corresponds to a sum (tagged union) type, and $\alpha \Rightarrow \beta$ corresponds to a function type. Modus ponens describes the type of an application, the deduction theorem corresponds to functional abstraction (lambda), and the other rules of inference correspond to data constructors or accessors.

The constructive tautologies turn out to be precisely the types of closed lambda terms, i.e., of the objects that can be constructed in a functional programming language without using ground terms.

Thus, for example, it is possible to write an ML program whose type is $(\alpha \vee \beta) \Rightarrow \{(\alpha \Rightarrow \gamma) \Rightarrow [(\beta \Rightarrow \gamma) \Rightarrow \gamma]\}$ (remember that `->` associates to the right)[2]:

```
- datatype ('a,'b) OR = First of 'a | Second of 'b;
datatype ('a,'b) OR = First of 'a | Second of 'b
- fun cases (First a) f g = f a
= |   cases (Second b) f g = g b;
val cases = fn : ('a,'b) OR -> ('a -> 'c) -> ('b -> 'c) -> 'c
```

Indeed, a closed program is a textual representation of a mental construction; therefore it can be, and should be, understood as a proof of the intuitionistic validity of its type.

4 Semantics

Early efforts to develop a satisfactory metamathematics for intuitionism were frustrated in large part because the early intuitionists were vehemently anti-formalist in their stance, and refused to accept that the organic activity of mathematics could be reduced to a mechanical set of rules. Of particular concern to philosophers of mathematics was their undefined notion of a *construction*, or equivalently, of an *algorithm* that transformed one kind of proof into another.

In this regard, the writings of the early intuitionists, especially Brower, were extraordinarily opaque, and seemed intended to obscure rather than enlighten[3].

Although intuitionists eventually conceded that the lambda-definable functions were all constructive, they did not grant that these are the only

[2]Note that while SML supports anonymous product and function types, it does not support anonymous sum types. Thus, the need for the `datatype` definition.

[3]Modern intuitionists, e.g., Errett Bishop, have been especially critical of their predecessors in this regard, so there's more going on here than just a logician's sour grapes.

valid proof transformations. The attitude was simple, but not especially useful for logicians: "I know a construction when I see one." Thus, any formula provable in this system is intuitionistically valid, but intuitionists reserve the possibility that there may be other formula, not provable in this system, which are nevertheless intuitionistically valid. As time as passed, and no such formula has been forthcoming, this possibility has seemed increasingly remote.

In the effort to pin the intuitionists down, a number of formal semantics for various intuitionistic calculi have been proposed. These semantics have been proposed in the attempt to pin the intuitionists down to a particular logical system, and in consequence typically attempt to formalize some intuitionist's explanation of their philosophy. This lead to the disquieting understanding that different intuitionists often arrive at, and understand, intuitionism in very different ways. The problem here is not just that it means that you can only hope to pin them down one at a time (although it does indeed mean that), but the very fact that the fact that so many different ways of thinking lead to the same theory is a powerful argument for the naturalness and importance of that theory. Intuitionism is not going to go away.

The most popular semantics is Kripke's, which is often described as "temporal epistemic", which is to say that it attempts to explain intuitionism in terms of how mathematicians acquire mathematical knowledge over time, and which finds antecedents in Brower's own writings.

Definition 1 *A* Kripke model *is a tuple* $(W, \leq, \models)$*, where* W *is a nonempty set of* worlds*,* $\leq$ *is a partial order on* W*, and* $\models$ *is a relation on* $W \times Var$*, where Var is the set of variables, such that for all* $u, v \in W$ *and all variables* α*, if* $u \leq v$*, and* $u \models \alpha$*, then* $v \models \alpha$*.*

As usual, we use $v \geq u$ as synonymous with $u \leq v$.

We extend the relation $\models$ to $W \times \text{Form}$, where Form denotes the propositional formula, but induction on structure:

1. $u \models \varphi \vee \psi$ if and only if $u \models \varphi$ or $u \models \psi$,

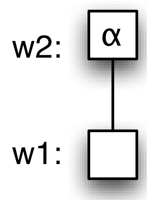
2. $u \models \varphi \wedge \psi$ if and only if $u \models \varphi$ and $u \models \psi$,

3. $u \models \varphi \Rightarrow \psi$ if and only if for all $v \geq u$, if $v \models \varphi$, then $v \models \psi$.

4. $u \models \neg\varphi$ if and only if for all $v \geq u$, $v \not\models \varphi$.

Kripke semantics are sound and complete for propositional intuitionistic logic, which is to say, φ is an intuitionistic tautology if and only if it holds at every world of every Kripke model.

As a general rule, we present Kripke models graphically, with nodes labeled according to the variables they model ($\models$), so called *ground truth*, and edges linking worlds related by $\leq$, with the greater worlds presented above the lower worlds. Thus,



is a simple Kripke model. Indeed, it is a particularly interesting Kripke model, because $\alpha \vee \neg\alpha$ (the law of the excluded middle), $\neg\neg\alpha \Rightarrow \alpha$ (double negation elimination), and $((\alpha \Rightarrow \beta) \Rightarrow \alpha) \Rightarrow \alpha$ (Peirce's law) all fail at w1.

5 Some Concluding Remarks

Intuitionism began as a reaction against nonconstructive methods, and in the early years, both intuitionists and classical mathematicians tended to view one another as apostate, with each side trying to call the others back to the "right" way to do mathematics.

Because both parties saw the issue as right versus wrong, neither was intellectually prepared to understand that there was considerable merit to the other point of view. If we can prove more classically than intuitionistically, we know more about a statement φ if we know it is intuitionistically valid than if it is (merely) classically valid.

The particular utility of this "more that we know" became apparent with the rise of computers and their application to practical problems, as intuitionistic proofs are simply programs—no more, but certainly no less—and you can do things with programs that you can't do with proofs.

6 Exercises

Problem 1 *Prove that* $\alpha \Rightarrow (\beta \Rightarrow \gamma) \vdash_I (\alpha \wedge \beta) \Rightarrow \gamma$.

Problem 2 *Show that* $\alpha \Rightarrow \beta \nvdash_I \neg\alpha \vee \beta$ *by demonstrating that there exists a Kripke model* $K = (W, \leq, \models)$ *and a world* $w \in W$ *such that* $w \models \alpha \Rightarrow \beta$, *but* $w \not\models \neg\alpha \vee \beta$.

Problem 3 *Show that world w1 in the simple Kripke model in Section 4 does not satisfy Peirce's law.*

Problem 4 *Consider the proof of triple negation reduction in these notes. This seems so closely related to double negation elimination that it seems as though it should be possible to turn a proof of one into a proof of the other. What is the impediment that prevents us from turning the proof of triple negation reduction into a proof of double negation elimination?*

Problem 5 *Show that Peirce's law follows intuitionistically from the law of the excluded middle, i.e., that* $\alpha \vee \neg\alpha \vdash_I [(\alpha \Rightarrow \beta) \Rightarrow \alpha] \Rightarrow \alpha$. *Hint: To apply proof by cases, it suffices to show* $\neg\alpha, (\alpha \Rightarrow \beta) \Rightarrow \alpha \vdash_I \alpha$.