Discrete Math - Homework 7 - Due Wednesday December 3rd

Prove all of your answers. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class.

1. Note: No late homeworks will be accepted after Dec. 3rd.

2. A *Hamiltonian path* in a graph G is an ordering of the vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$ such that $\{v_{i_j}, v_{i_{j+1}}\}$ is an edge for all $1 \leq j \leq n-1$. (v_{i_1} and v_{i_n} do not need to be adjacent this would give a Hamiltonian cycle.)

   A *tournament* is a directed graph such that for each pair of vertices u, v, exactly one of (u, v) and (v, u) is an edge of the graph.

   Prove that every tournament has a Hamiltonian path.

3. Define the compliment of a graph $G = (V, E)$ to be the graph $\bar{G} = (V, \bar{E})$. (Namely it has the complimentary set edges.) Prove that if G is a simple graph with at least 11 vertices, then either G or $\bar{G}$ is nonplanar.

4. Prove that if T is a stochastic matrix then T^k is stochastic for all positive integers k.

5. For any graph, we can form a Markov chain by letting the transition from vertex v to any of its neighbors be $1/deg(v)$. Prove that such a Markov chain is periodic if and only if the graph is bipartite.

6. Say you have n homework problems and you have already solved k of them. Everyday you pick one problem equally at random to look at. If you haven't already, you solve the problem. Let X_i represent the event that i problems are unsolved.

   (a) Prove that this process is a Markov chain with states X_i.
   (b) What is the transition matrix for this chain?
   (c) What states are recurrent, transient, periodic?

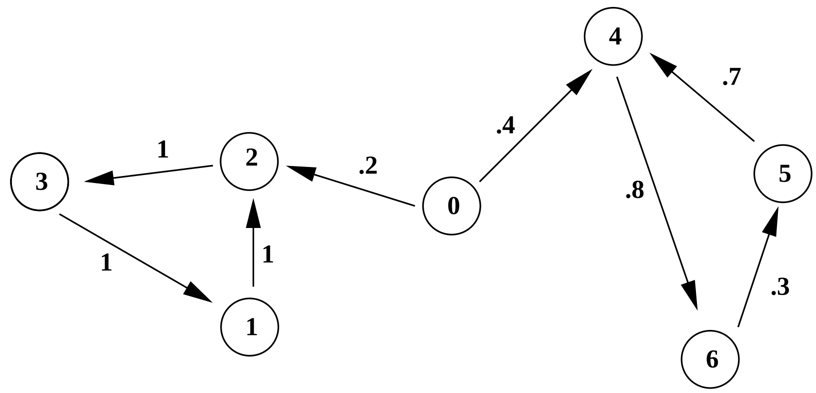

Figure 1: The Markov Chain for problem 6

7. Assume the Markov chain of Figure 1 is in state 0 just before the first step.
   (a) What states are recurrent, transient, periodic?
   (b) What is the probability that after 6 steps you are in state 3 ?
   (c) What is the probability that after 6 steps you are in state 5 ?
   (d) What is the expected number of steps before the process leaves state 0 ?
   (e) What is the probability that we never get to state 6 ?
   (f) For each ergodic recurrence class, find the stationary distribution.