Discrete Mathematics 271 sketch of solutions

2. Induct on the number of vertices n. For $n = 2$ it is obvious. Assume the claim holds for $k \leq n$ and G is a tournament on $n + 1$ vertices. Remove any vertex v leaving a graph on n vertices. This smaller graph has a Hamiltonian path by induction. Let v_i be the vertex of highest label such that (v_i, v) is an edge. Then (v, v_{i+1}) must also be an edge. Therefore we can create a Hamiltonian path on the full graph by visiting v between nodes v_i and v_{i+1}.

3. Second, together G and its complement must have all posisble edges on 11 vertices, which is $\binom{11}{2} = 55$ possible edges. So one must have at least 28 edges. Now Euler's formula show's that this graph can not be planar. Careful, what about connectedness of the graph?

5. Recall that a graph is bipartite iff it has no odd cycles. In forming the Markov chain given by degrees, we induce those cycles from the underlying graph and cycles between any pair of neighboring vertices. Namely, walk to a neighbor and then walk back. Thus any such chain will have an even cycle of length two. If the graph had no odd cycles, then the Markov chain will have all even cycles and be periodic. Otherwise for any odd cycle of length k in the chain from state s to s we can also walk from state s to s in $k + 1$ steps by visiting a neighbor.

6. (b) We have non-zero entries along the diagonal and on the first off diagonal. $T_{i,i} = \frac{n-i}{n}$ = the probability of picking a problem that is already solved. $T_{i,i-1} = \frac{i}{n}$ = the probability of picking a problem that is unsolved. Note the boundary cases. If $k = 0$ then on the first step you must transition out of the current state. When $i = 0$ there are no more unsolved problems and you stay at X_0 forever. (c) All states are transient expect for the last state X_0 which is recurrent and not periodic.

7.(a) Transient: 0, Recurrent: 1,2,3,4,5,6, Periodic: 1,2,3. Note that there is a non-zero probability of staying in place at each of 4,5, and 6.

(b) The only way to be in state 3 after 6 steps is to stay at state 0 on the first step and then move to state 2 on the second step. This has probability .4*.2.

(c) This is the $p_{0,5}$ entry of T^6. You can also recursively write the probability: To be in state 5 after 6 steps means you must be at either state 5 or state 6 after 5 steps... Nothing to special in this problem, just trying to appreciate the elegance of powers of the transition matrix.

*** Parts (d) and (e) actually deal with infinite spaces which we didn't address. I wanted to get you thinking about such things. You will not see this kind of problem on the final exam.

(d) With probability 1 we will eventually leave state 0. Furthermore if we walk to the right we will again with probability 1 eventually hit state 6 thus we only need to know the probability we walk to the the right when we leave state 0. This probability is $\frac{.4}{.2+.4} = \frac{2}{3}$.

(e) Let E be the expected number of steps before leaving state 0. Then $E = .6 + .4E$. Either we leave in one step or we stay in place and start over.

(f) For the collection of states 4,5,6, solve the equations:

$$\pi_4 = .2\pi_4 + .7\pi_5$$

$$\pi_5 = .3 * \pi_5 + .3\pi_6$$

$$\pi_6 = .7\pi_6 + .8\pi_4$$

$$\pi_4 + \pi_5 + \pi_6 = 1$$