Discrete Math - Practice Problems Solutions - 2005

1. Let $a = bq + r$ (a, b, q, r all positive integers). Prove that $\gcd(a, b) = \gcd(b, r)$.

   See Textbook Theorem 3.34

2. Find the remainder of $1! + 2! + \ldots + 99! + 100!$ when divided by 18.

   All terms after and including 6! are divisible by 18. $1+2+6+24+120 = 153$. $153 = (18 * 8 + 8)$. So the remainder is 8.

3. Give a combinatorial proof of the following identity:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

   The RHS counts the number of subsets of size n from a set of size $2n$. Now consider the initial set of size $2n$ divided in half. To pick a subset of size n we must pick k elements from the first half and $n-k$ from the second half. Hence the number of subsets equals $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$. But $\binom{n}{n-k} = \binom{n}{k}$ hence we have the LHS.

4. Give a closed form expression for

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \ldots.$$

   This counts the number of even subsets of a set of size n. We know this is half of all the subsets and hence equal to 2^{n-1}.

5. Count the number of 4 cycles in the complete bipartite graph $K_{n,n}$.

   To form a 4 cycle, we must choose 2 vertices from V_1 and 2 vertices from V_2. Furthermore, any such choice gives us a unique 4 cycle. The total number is then $\binom{n}{2}^2$.

6. Let G be a graph on n vertices that is not connected. What is the maximum number of edges that G can have?

   A graph on n vertices can have at most $\binom{n}{2}$ edges. In order to disconnect G we must remove *at least* $(n-1)$ edges. This leaves the graph which has two connected components, K_{n-1} and a single vertex. This graph has $\binom{n-1}{2}$ edges.

7. Suppose we color the *edges* of K_6 with two colors. Prove that no matter how we color the edges there will exist a monochromatic triangle (A triangle of only one color). *Hint* Consider the degree of any vertex.

   Any vertex v has degree 5 so at least 3 of the edges incident to v must have the same color. Let these three edges be vv_1, vv_2, and vv_3. Without loss of generality assume the color of these edges is red. Now if any of the edges v_iv_j $i, j \in \{1, 2, 3\}$ is colored red then we have a red triangle. If none of these edges is colored red, then they are all colored blue and form a blue triangle.

8. Let G be an arbitrary plane graph with v vertices, e edges, f regions, and m connected components. Prove that $v - e + f = m + 1$.

   Each connected component C_i of G satisfies Euler's formula: $v_i - e_i + f_i = 2$. If we take the sum over all components we get: $v - e + f = 2m$. However we have overcounted the outer region $m - 1$ times. Hence we get: $v - e + = 2m - (m - 1) = m + 1$.

9. Does there exist a graph on 6 vertices which is (a) Hamiltonian but not Eulerian (b) Eulerian but not Hamiltonian?

   The answer is yes for both parts. Possible examples:(a) K_4 (b) See Figure.

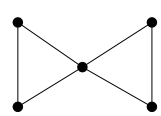

Figure 1: Eulerian but not Hamiltonian

10. Suppose you roll a fair 6 sided die 100 times. Let X be the number of times two consecutive rolls result in the same number. (a) What is the $E[X]$? (b) What is the $V(X)$?

    Define indicator random variables X_i which are equal to 1 when roll i is the same as roll $i + 1$ and 0 otherwise. X is then the sum of X_1 through X_{99}. $E[X_i] = \frac{1}{6}$ so $E[X] = \frac{99}{6}$. $V(X_i) = \frac{5}{36}$ and the X_is are independent so $V(X) = \frac{(99)5}{36}$.

11. Let X and Y be independent random variables. Prove that: $V(X+Y) = V(X) + V(Y)$.

    $V(X+Y) = E[(X+Y)^2] - E[(X+Y)]^2 = E[X^2+2XY+Y^2] - (E[X]+E[Y])^2 = E[X^2] + E[2XY] + E[Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2$. Because X and Y are indpendent, $E[XY] = E[X]E[Y]$. Hence we have $E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 = V(X) + V(Y)$.

12. Suppose a multiple choice problem has n possible answers. You guess an answer equally likely at random. If you get it wrong, you pick an answer equally at random from the remaining choices. You continue until you have guessed the right answer, each time only guessing from answers you have not tried yet. What is the expected number of guesses you take until you get the right answer?

    Let X be the number of guesses until you get the answer right. $p(X = k) = \frac{n-1}{n}\frac{n-2}{n-1}\cdots\frac{n-k}{n-k+1}\frac{1}{n-k} = \frac{1}{n}$. Therefore $E[X] = \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}$.

13. Let a_n be the sequence for the Fibonacci numbers: $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Find a simple closed expression for the ordinary generating function $F(x) = \sum_{n\geq 0} a_n x^n$.

    $\sum_{n\geq 0} a_n x^n = 0 + x + \sum_{n\geq 2}(a_{n-1} + a_{n-2})x^n = x + x\sum_{n\geq 2} a_{n-1}x^{n-1} + x^2 \sum_{n\geq 2} a_{n-2}x^{n-2} = x + xf(x) + x^2 f(x)$.

    $f(x) = x + xf(x) + x^2 f(x) \Rightarrow f(x) = \frac{x}{1-x-x^2}$.

14. If $f(x)$ is the ordinary generating function for the sequence a_n, and $b_n = na_n$ what is the generating function for b_n in terms of $f(x)$?

    Let $g(x)$ be the ordinary generating function for b_n then $g(x) = xf'(x)$.

15. Let $a_n, b_n \to \infty$. Show if $a_n = \Theta(b_n)$ then ln $a_n \sim$ ln b_n.

    By definition there exist positive constants c_1 and c_2 such that for all sufficiently large n,

    $$c_1 b_n \leq a_n \leq c_2 b_n.$$

    Taking the logarithms of each side,

    $$\ln c_1 + \ln b_n \leq \ln a_n \leq \ln c_2 + \ln b_n.$$

    Let us divide each side by $\ln b_n$:

    $$1 + (\ln c_1/\ln b_n) \leq \ln a_n/\ln b_n \leq 1 + (\ln c_2/\ln b_n).$$

Now $b_n \to \infty$ and therefore $\ln c_i / \ln b_n \to 0$. So both extreme sides of the double inequality approach 1 and hence, by the sandwich principle, the quantity sandwiched between them, $\ln a_n / \ln b_n$, also approaches 1, proving that $\ln a_n \sim \ln b_n$.